Please put problem 1 on answer sheet 1

1. (a) Suppose that $\bar{v}$ is an eigenvector for both matrices A and B. For A the eigenvalue is λ_A and for B the eigenvalue is λ_B. Show that the matrices AB and BA have an eigenvalue in common.

   (b) Use the characteristic polynomial to find the eigenvalues of $\begin{bmatrix} 2 & 7 \\ 2 & -3 \end{bmatrix}$.

Please put problem 2 on answer sheet 2

2. (a) Give an example of each of the following. No explanation is required.
   i. A 2×2 matrix with no eigenvalues.
   ii. A 2×2 matrix with a single eigenvalue for which the eigenspace is 2-dim.

   (b) Use the Gram-Schmidt process to find an orthonormal basis for

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

Please put problem 3 on answer sheet 3

3. Define

$$A = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}.$$

   (a) Find the characteristic polynomials for A and B

   (b) Given that B has eigenvalues $\lambda = 1$ and $\lambda = 3$, diagonalize B.

   (c) Write B^5 as the product of three matrices, the third being the inverse of the first.

Please put problem 4 on answer sheet 4

4. (a) Suppose that A and B are diagonalizable and have the same eigenvectors. Prove that $A + B$ is diagonalizable.

   (b) Given that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is an orthogonal subset of $\mathbb{R}^3$, prove that $\{\bar{u}_1+\bar{u}_2, \bar{u}_1-\bar{u}_2, -\bar{u}_3\}$ is a basis for $\mathbb{R}^3$.

   (c) Find all least-squares solutions to the matrix equation $\begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \bar{x} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

Please put problem 5 on answer sheet 5

5. Define the quadratic form $Q(\bar{x}) = -5x_1^2 + 4x_1x_2 - 2x_2^2$.

   (a) Classify $Q(\bar{x})$ as positive definite, negative definite or indefinite and justify.

   (b) Find a change of variables which converts $Q(\bar{x})$ into another quadratic form (call it $R(\bar{y})$) with no cross-product term. Indicate the change of coordinates matrix P, the new quadratic form $R(\bar{y})$ and the matrix for this new form.

   (c) Use your R to find $Q\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ by way of your change of variables.
