# MATH 240 – Spring 2013 – Final Exam

CALCULATORS ARE NOT ALLOWED
TURN OFF ALL ELECTRONIC DEVICES

You may assume given matrix equations are well defined (i.e. the matrix sizes are compatible).

1. (a) (25 points) The following matrices are row equivalent:

$$W = \begin{pmatrix} 4 & 2 & -1 & 4 & 7 \\ 3 & 1 & 0 & 2 & 3 \\ 1 & 1 & -1 & 2 & 4 \\ 2 & 2 & -3 & 0 & -1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 8 & 18 \\ 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) [7 pts] Write down a basis for the row space of $W$.
**Solution.**
The first three rows of $R$ (not $W$) are a basis for the row space of $W$.

(b) [7 pts] Write down a basis for the column space of $W$.
**Solution.**
The first three columns of $W$ (not $R$) are a basis for the row space of $W$.

(c) [8 pts] Write down a basis for the null space of $W$.
**Solution.**
The null space has a basis of two vectors: $\begin{pmatrix} 5 \\ -18 \\ -9 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -8 \\ -4 \\ 1 \\ 0 \end{pmatrix}$.

(d) [3 pts] What is the rank of $W$?
**Solution.**
The rank of $W$ is 3.

(b) (15 points) TRUE or FALSE? No justification required.

i. The set of vectors $(x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that $x_1x_2x_3 = 0$ is a subspace of $\mathbb{R}^3$.
**FALSE.**

ii. The set of polynomials $\{p_1, p_2, p_3, p_4\} = \{2+t,\ 3t+t^2,\ -7+t+4t^2,\ -1+9t+8t^2\}$ is linearly independent.
**FALSE.**
(Four vectors in the three dimensional space of quadratic polynomials cannot be linearly independent.)

iii. Let $V$ be the vector space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and define the function $T$ from $V$ to $\mathbb{R}$ by the rule $T(f) = f(2)$. Then $T$ is a linear transformation.
**TRUE.**

2. (a) (10 points) Suppose $A, B, C$ are invertible $3\times 3$ matrices and as a partitioned matrix (block matrix) the $6 \times 6$ matrix $U$ has the form $U = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$.

Find the block matrix form for $U^{-1}$.

**Solution.**

$$U^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix} .$$

(b) (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates a vector counterclockwise around the origin by $5\pi/4$ radians and then multiplies its length by 3. Find the matrix $A$ such that $T(x) = Ax$ for all $x$.

**Solution.**

$$A = 3\begin{pmatrix} \cos(5\pi/4) & -\sin(5\pi/4) \\ \sin(5\pi/4) & \cos(5\pi/4) \end{pmatrix} = \begin{pmatrix} -3/\sqrt{2} & 3/\sqrt{2} \\ -3/\sqrt{2} & -3/\sqrt{2} \end{pmatrix} .$$

(c) (15 points) Below, $U$ is an $n \times n$ orthogonal matrix.

i. Define what it means for $U$ to be an orthogonal matrix.

**Solution.**

$U$ is invertible with $U^{-1} = U^T$ (where $U^T$ denotes transpose of $U$).

ii. Suppose $z$ is a vector in $\mathbb{R}^n$. Prove that $||Uz|| = ||z||$ .

**Solution.**

With $< x.y >$ used to denote the dot product of $x$ and $y$:

$$\begin{aligned} ||Uz||^2 &= < Uz, Uz > = (Uz)^T(Uz) = (z^TU^T)(Uz) = z^T(U^TU)z \\ &= z^T(I)z = z^Tz = ||z||^2 . \end{aligned}$$

Since $||Uz||^2 = ||z||^2$, it follows that $||Uz|| = ||z||$ .

iii. Suppose $x$ and $y$ are vectors in $\mathbb{R}^n$. Prove that $\text{dist}(Ux, Uy) = \text{dist}(x, y)$ (where e.g. $\text{dist}(x, y)$ denotes the distance from $x$ to $y$).

**Solution.**

$$\begin{aligned} \text{dist}(Ux, Uy) &= ||Ux - Uy|| \\ &= ||U(x - y)|| \\ &= ||x - y|| \quad \text{by part ii} . \end{aligned}$$

3. Define

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix} .$$

(a) (25 points) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$, or prove that no such matrices exist.

**Solution.**

$$D = \begin{pmatrix} 7 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix} .$$

(b) (15 points) TRUE or FALSE? No justification required.

i. If $A$ is a matrix, then $A^T A$ is a symmetric matrix.
**TRUE.**
$((A^T A)^T = A^T (A^T)^T = A^T A$ . )

ii. For a matrix $A$ and an invertible matrix $U$, $\text{rank}(UA) = \text{rank}(A)$.
**TRUE.**

iii. If $A = [2\ 2\ ;\ 2\ 2]$ (in MATLAB notation) and $S = \{Ax : ||x|| \leq 1\}$, then the area of $S$ equals $4\pi$.
**FALSE.**
(Note $\det(A) = 0$ .)

4. (a) (10 points) For an $m \times n$ matrix $A$, write the equation relating the rank of $A$ and the dimension of its null space.
**Solution.**
$\text{rank}(A) + \dim(\text{nul}A) = n$ .

(b) (10 points) Suppose that $(t-2)^2(t-1)(t+2)t$ is the characteristic polynomial of a matrix $M$, and the null space of $M - 2I$ has dimension 2.
Which of (a), (b) or (c) below is the correct statement? (No justification required.)
(a) $M$ must be diagonalizable.
(b) $M$ might be diagonalizable.
(c) $M$ cannot be diagonalizable.
**Solution.**
(a) $M$ must be diagonalizable.

(c) (25 points) Find the numbers $\beta_0, \beta_1$ such that the line $x_2 = \beta_0 + \beta_1 x_1$ is the least squares best fit to the data points $(0,4), (1,3), (3,2), (4,1)$.
**Solution.**
We solve:

$$\begin{aligned}
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} &= \begin{pmatrix} 10 \\ 13 \end{pmatrix} \\
\begin{pmatrix} 4 & 8 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} &= \begin{pmatrix} 10 \\ -7 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} &= \begin{pmatrix} 2.5 \\ -.7 \end{pmatrix} \\
\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} &= \begin{pmatrix} 3.9 \\ -.7 \end{pmatrix} .
\end{aligned}$$

5. (a) (10 points) TRUE or FALSE?

   i. If $A$ is a $2 \times 2$ matrix with rank 2, then $\{Ax : ||x|| = 1\}$ is an ellipse.
   **TRUE.**

   ii. The function $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 3x + 1$ is a linear transformation.
   **FALSE.**

   (b) (5 points) What is the $3 \times 3$ elementary matrix $E$ such that $EA$ is obtained by adding 5 times row 3 of $A$ to row 1 of $A$?
   **Solution.**
   $E = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

   (c) (10 points) Give the definition of the dimension of a finite dimensional vector space.
   **Solution.**
   The dimension is the number of vectors in a basis for $V$.

   (d) (15 points) For the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, what is the maximum length of a vector $Bx$, under the constraint that $||x|| = 1$ ?
   **Solution.**
   Let $M$ denote $B^T B$; then

$$||Bx||^2 \; = < Bx, Bx > = \; x^T(B^T B)x \; = \; x^T(M)x$$

   and

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} .$$

   The matrix $M$ is symmetric. We know then that the maximum value of $x^T M x$ subject to $||x||^2 = 1$ is the largest eigenvalue of $M$. The characteristic polynomial of $M$ is $t^2 - 6t + 1$. Its roots are

$$\frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm \sqrt{8} .$$

   The largest eigenvalue of $M$ is then $3 + \sqrt{8}$, and given the constraint $||x|| = 1$ this is the maximum value of $x^T M x = ||Bx||^2$. Therfore the largest value of $||Bx||$ given $||x|| = 1$ is

$$\sqrt{3 + \sqrt{8}} \quad .$$