MATH 240 – Spring 2013 – Exam 1

CALCULATORS ARE NOT ALLOWED
TURN OFF AND STORE ALL ELECTRONIC DEVICES
*** THERE ARE QUESTIONS ON BOTH SIDES OF THIS PAPER ***

There are five questions. Answer each question on a separate sheet of paper. Use the back side if necessary.
On each sheet, put your name, your section TA's name and your section meeting time.
You may assume given matrix equations are well defined (i.e. the matrix sizes are compatible).

1. (20 points) The following two 4×6 matrices can be obtained from each other by performing elementary row operations.

$$A = \begin{pmatrix} 4 & -24 & -3 & 1 & 3015 & 82 \\ 2 & -12 & 4 & -8 & 105 & 90 \\ 6 & -36 & 1 & -5 & 3618 & 220 \\ 12 & -72 & 2 & -13 & 7839 & 368 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -4 & 0 & 0 & 5 & 49 \\ 0 & 0 & 1 & 0 & -4 & 46 \\ 0 & 0 & 0 & 1 & -2 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Suppose A is the augmented matrix of a system of linear equations which would be written in matrix form as $Cx = b$ (where C is the coefficient matrix of the system).

i. (4 pts.) If the variables are denoted as $x_1, x_2, \ldots$, which of the variables are the free variables?

ii. (5 pts.) Write the basic variables as functions of the free variables, giving a new system with the same solution set.

iii. (4 pts.) Write the general solution x of $Cx = b$ in the standard parametrized form: x equals a numerical vector plus a linear combination of numerical vectors, where the coefficients of the linear combination are the free variables.
(For example, if x_1 and x_2 were the free variables, then an answer would have the form $x = u + x_1 v + x_2 w$, where u, v, w are numerical column vectors.)

(b) (3 pts.) Given A, what MATLAB command produces the matrix R?

(c) (4 pts.) Answer TRUE or FALSE (no explanation needed): Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be the linear transformation defined by the rule $T(x) = Ax$, for the matrix A above. Then T is onto.

2. (20 points)

(a) (6 pts.) Suppose M is a 20×20 matrix, and $M = \begin{pmatrix} A & 0 \\ X & B \end{pmatrix}$ is a block (partitioned) matrix, in which the matrices A, B are invertible and 0 is a matrix of appropriate size with every entry zero. Write a formula for M^{-1} as a block matrix.

(b) (6 pts.) Write the 3×3 matrix E such that EA is the matrix obtained from A by subtracting row 1 from row 3, and leaving rows 1 and 2 unchanged.

(c) For each of the following, answer TRUE or FALSE. No justification needed.

i. (4 pts.) Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 3x + 3$. Then T is a linear transformation.

ii. (4 pts.) Every square matrix is a product of elementary matrices.

3. (20 points)

(a) For the following maps T_i : find the matrix A_i such that $T_i(x) = A_i x$ for all x.

i. (4 pts.) T_1 is the map which takes a vector x in $\mathbb{R}^2$ and rotates it counterclockwise around the origin through an angle $3\pi/4$.

ii. (4 pts.) T_2 is the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects a vector through the line $x_1 + x_2 = 0$.

iii. (4 pts.) T_3 is the map defined by $T_3(x) = T_2(T_1(x))$.

(b) For the following, answer TRUE or FALSE. No explanation required.

i. (4 pts.) It is impossible for 7 vectors in $\mathbb{R}^{10}$ to span $\mathbb{R}^{10}$.

ii. (4 pts.) If A, B are matrices and column 1 of B is a zero column, then column 1 of AB is a zero column.

4. (20 points)

(a) ($T : x \mapsto y$ means that the function T takes the input x to the output y.)
Suppose T is a linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 2 \\ 6 \end{smallmatrix}\right)$ and $T : \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right)$. Let A denote the standard matrix for T (this is the matrix A such that $T(x) = Ax$ for all x).

i. (4 pts.) The input-output data can be formulated as a matrix equation $AB = C$, with $B = \left(\begin{smallmatrix} 4 & 1 \\ 2 & 1 \end{smallmatrix}\right)$.
What is the matrix C?

ii. (8 pts.) Compute A.

(b) For the following, answer TRUE or FALSE. No explanation required.

i. (4 pts.) If v_1, v_2 and v_3 are vectors in $\mathbb{R}^3$, and none of them is a scalar multiple of one of the others, then $\{v_1, v_2, v_3\}$ is linearly independent.

ii. (4 pts.) If A and B are invertible 5×5 matrices, then AB is an invertible matrix.

5. (20 points)

(a) (12 pts.) Which of the following matrices are invertible? No justification required.

$$A = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 8 & -3 & 0 \\ 0 & 6 & 7 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & 0 & 8 \\ -1 & 7 & 6 & 0 \\ 2.1 & 0 & 4 & 4 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 4 & 1 & 3 & 8 \\ 7 & 1 & 0 & 6 & 0 \\ 0 & 5 & 2 & 1 & 1 \end{pmatrix}$$

(b) For the following, answer TRUE or FALSE. No explanation required.

i. (4 pts.) If A, B, C are matrices such that $AB = AC$, then $B = C$.

ii. (4 pts.) Suppose A is an $m \times n$ matrix, and b is a column vector in $\mathbb{R}^m$, and the vectors u and v in $\mathbb{R}^n$ are solutions to the equation $Ax = b$.
Then there is a vector w which is a solution to $Ax = 0$ such that $v = u + w$.