MATLAB Project 2: MATH240, Spring 2013

1. **Method**

   This page is more information which can be helpful for your MATLAB work, including some new commands. You're responsible for knowing what's been done before. The project starts on the next page. The previous guidelines on format still hold.

2. **New Commands**

   (a) Powers of matrices can be done in the obvious way. If `A` is a matrix then we can do:

   ```
   >> A^2
   ```

   or:

   ```
   >> A^17
   ```

   You cannot however do the inverse of a matrix this way:

   ```
   >> A^-1
   ```

   This will not work.

   (b) Matrix multiplication can be done with:

   ```
   >> A*B
   ```

   This also takes care of matrices times vectors. For example if

   ```
   >> A=[1 2 3;4 5 6;7 8 9]
   >> v=[-2 ; 1 ; 3]
   ```

   then we can do the product $A\bar{v}$ with

   ```
   >> A*v
   ```

   (c) The determinant of a matrix is done with the `det` command. This can be done either on a given matrix:

   ```
   >> A=[1 2 3;4 5 6;7 8 9]
   >> det(A)
   ```

   or inline:

   ```
   >> det([1 2 3;4 5 6;7 8 9])
   ```

   (d) The inverse of a matrix can be done with the `inv` command:

   ```
   >> inv(A)
   ```

   or inline as with `det`.

3. **Combining Functions**

   MATLAB allows you to combine commands easily. For example, suppose we wish to solve the system

   $$\begin{aligned} 2x_1 + x_2 &= 0.5 \\ -3x_1 - 5x_2 &= 10 \end{aligned}$$

   We can do this with a single MATLAB command line entry with an inverse matrix:

   ```
   >> inv([2 1;-3 -5])*[0.5;10]
   ```

MATH 240 Spring 2013 MATLAB Project 2

Directions: A paper (not emailed) version of the completed project should be handed in to your TA when due (or earlier). This can be a printout of the command window of a clean final session, as in the format guidelines described for Project 1.
ALSO ALLOWED: you can build up your mfile, and produce the final document by "publishing" it in MATLAB.
Any question marked with a star $\star$ requires an answer which is not explicitly a MATLAB result. Answer in a comment using % (as in Project 1).
You can work in groups of 1,2 or 3, with anyone in the course. List all members of your group at the beginning of your project.
Everyone in a group should type up the project individually, and hand it in to his/her TA. . You can't learn MATLAB without typing. Ideally you work together, working on your own laptops. Unless TAs choose otherwise, one project from a group will be graded and everyone in the group will get that grade.

More details:

1. Before each question, type `clear` to clear out the memory.
2. Do questions 1-4 in format rat (do `format rat`).
3. In questions 5,6,7 work in `format long` or `format rat` as appropriate. "long" exhibits more digits of precision. "rat" gives you rational expressions (fractions) rather than decimals. "rat" can be better for seeing identities and fractions, "long" may be better for seeing precision.
4. To return to the standard format from (say) format rat, enter `format short`.

On to the project problems:

---

1. Recall, an indexed set of vectors $v_1, \ldots, v_k$ is linearly dependent iff one of the vectors can be written as a linear combination of other vectors in the set. Unless the first vector v_1 is the zero vector, the indexed set is linearly dependent iff one of the vectors can be written as a linear combination of *previous* vectors in the set.

   To do this for the following set $\mathcal{S} = \{v_1, v_2, v_3, v_4, v_5\}$ of vectors,

   $$\mathcal{S} \;=\; \left\{ \begin{bmatrix} 2 \\ -2 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ 24 \\ 16 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -8 \\ 48 \end{bmatrix}, \begin{bmatrix} 24 \\ -10 \\ 96 \\ 0 \end{bmatrix}, \begin{bmatrix} 59 \\ -14 \\ 128 \\ 70 \end{bmatrix} \right\} ,$$

   proceed as follows. (The following MATLAB might be useful: if A is a matrix, then e.g. $A(:,3)$ denotes the column vector which is column 3 of the matrix.)

   (a) Define an appropriate matrix A.

   (b) Compute $R = \mathrm{rref}(A)$.

   (c) $\star$ By inspection from R: state the largest k such that v_k is a linear combination of the vectors v_i with $i < k$, $v_k = c_1 v_1 + \cdots + c_{k-1} v_{k-1}$. Also by inspection, type in a column vector $c = [c_1; c_2; \;\cdots\; ; c_{k-1}]$ which gives that linear combination.

   (d) (i) Doublecheck by computing the linear combination you gave.
   (ii) Also compute the vector Ad, where d is the vector whose first $k-1$ entries are those of c, and whose other entries are zero.
   (iii)$\star$ Explain the relationship of the results of the computations for (i) and (ii).

2. Suppose a linear transformation T has the property that $T(7,5) = (6.4,3)$ and $T(8,5) = (1,5/2)$. (Recall e.g. $(6.4,3)$ is Lay's notation for a column vector, written $[6.4;3]$ in MATLAB.)
Let A denote the matrix such that $T(x) = Ax$ for all x.

   (a) The data tell you there are matrices B and C such that $A * B = C$. Define B and C.
   (b) Using $\mathrm{inv}(B)$, C and matrix multiplication, compute A.
   (c) Check that you have the correct A by computing in MATLAB $A * [7;5]$ and $A * [8,5]$.

3. (a) Define
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$
   (b) Display the inverse using `inv`. Use the format which displays entries as fractions (not decimals).
   (c) (Now we let MATLAB do the work for the row reduction algorithm described in the text for finding A^{-1}.)
(i) Define the 3×6 matrix B which is $[\ A \mid I\ ]$.
(ii) Define $C = \mathrm{rref}(B)$.
(The left half of C is I and the right half of C is A^{-1})
(iii) Exhibit A^{-1} with the command $C(:,4:6)$
(this is the submatrix of C in columns 4 through 6, which MATLAB will exhibit).

4. (a) Use MATLAB to compute the determinants of the following two matrices.
$$A = \begin{pmatrix} 5 & 90 & 89 & -2 \\ 0 & -1 & 0 & 18 \\ 0 & 0 & -5 & 17 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 3 & -1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$
   (b) Use MATLAB to compute the matrix $C = AB$ and also $\det(C)$.
   (c) $\star$ What is the general fact for which these computations give an example?

5. Consider the set of functions
$$\left\{1, \cos t, \cos^2 t, \cos^3 t\right\}.$$
We would like to show that this is a linearly independent set of functions. This means that the equation
$$x_1(1) + x_2 \cos t + x_3 \cos^2 t + x_4 \cos^3 t = 0 \qquad (*)$$
has only the trivial solution: it is only true when each of the x_i is the number zero. The thing to keep in mind is that this equality is *functional*: for the equation to be true for a given set of numbers $x_1, ..., x_4$, it must be true for *all* values of t. (This kind of independence is important for understanding solutions of differential equations, e.g. in MATH 246.)

   (a) Each substitution of a number for t in the displayed equation produces a numerical equation. Using $t = 0, 0.1, 0.2$ and 0.3, you get four equations. Define the coefficient matrix A for this system, for MATLAB to display. (If for example you type cos(.1) into a matrix, MATLAB will know how to compute it as a number.)
   (b) If A is invertible, then $Ax = 0$ has no nontrivial solution. This implies the equation (*) has no nontrivial solution (because there is no nontrivial solution which works for $t = 0, 0.1, 0.2, 0.3$, let alone for all the other inputs). Compute `rref`(A). Compute $\det(A)$.
   (c) $\star$ Very briefly explain why each of the last two computations show A is invertible.
   (d) It is reasonable to be suspicious of the very small value of $\det(A)$ in the last step – could this be roundoff error, with the actual $\det(A)$ being zero? Do a check by repeating the computation with the more spread out inputs $t = 0, .2, .5, 1$, to see $\det(A)$ large enough to eliminate that suspicion.

6. Often a functional relation which holds at some fairly random inputs will hold at all inputs, and such a relation can be discovered numerically, and then verified with a proof. (Proofs are a lot easier when you are almost certain the claim you are proving is correct.) We'll start with an easy example.
We consider whether the set of functions

$$\{1, \cos^2 t, \sin^2 t\}$$

is a linearly independent set, that is, whether there are numbers x_1, x_2, x_3 such that for all t,

$$x_1(1) + x_2 \cos^2 t + x_3 \sin^2 t = 0 \ .$$

   (a) Use an approach with MATLAB similar to the above, defining a coefficient matrix A at inputs $t = 0, 0.1, 0.2$ for a corresponding system of equations.

   (b) Exhibit $R = \mathrm{rref}(A)$.

   (c) $\star$ Write down a nontrivial solution x to $Rx = 0$ (and hence to $Ax = 0$) obtained from the last step. Here $x = [x_1; x_2; x_3]$ such that $x_1(1) + x_2 \cos^2 t + x_3 \sin^2 t = 0$ if t is $0, 0.1$ or 0.2 . Choose your solution x with integer entries.

   (d) For the numerical vector x you discovered above, see if it holds (perhaps up to a tiny error from roundoff) for the "random" number 1.7392.

   (e) $\star$ Quote a basic trig identity which explains why your linear combination vanishes at every t.

7. Now, a more interesting example. We investigate whether the set of functions $\{\sin(t), \sin(3t), \sin^3(t)\}$ is linearly independent.

   (a) Proceeding as before, define a suitable matrix A and $B = \mathrm{rref}(A)$ to find a nontrivial linear relation for the set of functions $\sin(3t), \sin(t), \sin^3(t)$ which holds for the inputs $0, 0.2, 0.4$.

   (b) $\star$ Write down integers c_1, c_2, c_3 such that $c_1 \sin(3t) + c_2 \sin(t) + c_3 \sin^3(t) = 0$ at the inputs $t = 0, 0.2, 0.4$ (perhaps up to a tiny roundoff error). Compute the value of your linear combination at another random input (say $t = 5.621$) and see if it vanishes there. Switch to format long to see the size. Try another inputs of your choice.

   (c) $\star$ Is it reasonable to suspect your nonzero values, being so small at several inputs, are roundoff error?

   (d) $\star$ (Purely optional – for those interested. This is not for grading.)
   To be sure the identity holds for all inputs t, you can give a proof.

   Remember Euler's identity for complex numbers: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
   So, $e^{i3t} = \cos(3t) + i\sin(3t)$.
   Next, compute e^{i3t} in another way. Using the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we find

$$\begin{aligned} e^{i3t} &= (e^{it})^3 = \Big(\cos(t) + i\sin(t)\Big)^3 \\ &= [\cos(t)]^3 + 3[\cos^2(t)][i\sin(t)] + 3[\cos(t)][i\sin(t)]^2 + [i\sin(t)]^3 \\ &= \Big(\cos^3(t) - 3(\cos(t))(\sin^2(t))\Big) + i\Big(3\cos^2(t)\sin(t) - \sin^3(t)\Big) \ . \end{aligned}$$

   The real and imaginary parts of the two ways of computing e^{i3t} must match. There is one more step: you can replace the $\cos^2(3t)$ with $1 - \sin^2(3t)$ and simplify.

8. (You will want to be back in format short .)

(a) Symbolically define the variables `a` through `i`, using the command
syms a b c d e f g h i

(b) Define matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad , \quad B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad .$$

Find the determinants of A^k and B^k for $k = 1, 2, 3, 6$.

(c) $\star$ Record how long MATLAB takes to compute $\det(B^{10})$ (note start and end times and subtract). Record second monomial in the polynomial MATLAB computes to be $\det(B^{10})$.
(DO NOT execute this command in the output you turn in – save the trees. The point of this computation and the next is to emphasize that determinant computations from the definition can easily become very slow and unwieldy.)

(d) (This is a discussion explaining the next part – nothing to turn in.)

Let us see why MATLAB is slow when it is commanded to compute $\det(B^{10})$ above symbolically ("symbolically" means exact computation of the polynomial – no numerical simplifications).

The determinant of a 3×3 matrix is a polynomial function of the entries, a sum of six monomials. Each of these six determinant monomials is a product of three entries of the matrix. For the 3×3 matrix B^k, a matrix entry is a sum of 3^{k-1} monomials. Plugging these into the determinant polynomial we have a sum of six polynomials, each being a sum of $(3^{k-1})(3^{k-1})(3^{k-1}) = 3^{3k-3}$ monomials.

So, for the symbolic $\det(B^k)$, poor MATLAB has to compute $6(3^{3k-3})$ monomials.

(e) Have MATLAB compute the value of $6(3^{3k-3})$ for $k = 1, 2, 5, 10$.

(f) Let $C = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 20 & 5 \\ 1 & 2 & 1/2 \end{bmatrix}$. Compute $\det(C)$.

(g) $\star$ For this matrix C and a positive integer n, what is the correct value of $\det(C^n)$?

(h) For $n = 5, 6, 8, 10, 12$, execute the MATLAB command $\det(C^n) * (1/10)^n$.

(These matrices C^n get quite large as n increases. Computations are much faster than for the symbolic case, but, still, one must be careful about computations on large matrices. With accumulated error, they can go wrong, as you see.)