Following Justin's Guide to MATLAB in MATH240 - Part 3

1. **Method**

   You may want to review the first two guides whilst reading this one; the assumption is that you are comfortable with all those commands though not all are necessary.

2. **New Commands**

   (a) Rank(A) will compute the rank of a matrix A.

   (b) Eigenvalues can be found easily. If `A` is a matrix then:

   ```
   >> eig(A)
   ```

   will return the eigenvalues. Note that it will return complex eigenvalues too, which we're not so concerned about. So keep an i open for those.

   (c) However the characteristic polynomial is interesting in its own right. To begin with note the useful command `eye(n)` which returns the $n \times n$ identity (eye-dentity?) matrix:

   ```
   >> eye(5)
   ```

   (d) So now let use `L` for λ and if we have a matrix like:

   ```
   >> A=[8 -10 -5;2 17 2;-9 -18 4]
   ```

   we can symbolically define `L`:

   ```
   >> syms L
   ```

   and then:

   ```
   >> det(A-L*eye(3))
   ```

   to get the characteristic polynomial for A.

   (e) We can solve it using `solve`. One useful fact is that `solve` will assume the expression equals 0 unless specified and will solve for the single variable. Therefore we can do:

   ```
   >> solve( det( L*eye(3) - A ) )
   ```

   to get the solutions to the characteristic equation.

   (f) Of course if we have an eigenvalue λ we can use `rref` on an augmented matrix $[A - \lambda I|\bar{0}]$ to lead us to the eigenvectors.

   (g) Even better: MATLAB can do everything in one go. If you recall from class, *diagonalizing* a matrix A means finding a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$. The diagonal matrix D contains the eigenvalues along the diagonal and the matrix P contains eigenvectors as colums, with column i of P corresponding to the eigenvalue in column i of D.

   To do this we use the `eig` command again but demand different output. The format is:

   ```
   >> [P,D]=eig(A)
   ```

   which assigns `P` and `D` for A, if possible. If it's not possible MATLAB returns very strange-looking output.

MATH 240 Spring 2013 MATLAB Project 3

Directions: Guidlines on format and collaboration are as before.

As before, a question part marked with a star $\star$ indicates the answer should be typed into your output as a comment – the question isn't asking for MATLAB output.

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1. Let A be the matrix $\begin{pmatrix} 1 & -3 & 7 \\ 2 & 5 & 6 \\ 7 & 1 & 33 \end{pmatrix}$.
   (a) Compute rref(A) and rank(A)
   (b) $\star$ What are the pivot positions of A?
   (c) $\star$ For a general $m \times n$ matrix B with k pivot positions,
   what are $\dim(\text{nul}B), \text{rank}(B)$ and $\dim(\text{range}B)$ in terms of k, m, n?
   (We use Lay's terminology for range: it is the space of outputs, not necessarily the codomain.)

2. Let $[x]_{\mathcal{B}}$ denote the coordinate vector of x with respect to a basis $\mathcal{B}$.
   For bases $\mathcal{B}$ and $\mathcal{C}$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ denotes the matrix P such that $P[x]_{\mathcal{B}} = [x]_{\mathcal{C}}$.
   (P is the change of coordinates matrix.) The following are bases for the vector space $\mathbb{P}_3$:

$$\begin{aligned} \mathcal{E} &= \{1, t, t^2, t^3\} \quad , \\ \mathcal{B} &= \{1, 2-2t, 2-t-t^2, 1+2t+t^3\} \quad , \qquad \text{and} \\ \mathcal{C} &= \{1+2t+t^3, 2-t, 3t-4t^2+t^3, t\} \quad . \end{aligned}$$

   (a) Let $\{b_1, b_2, b_3, b_4\}$ denote the vectors of $\mathcal{B}$. Exhibit the 4×4 matrix B for which column i is $[b_i]_{\mathcal{E}}$.
   (b) Let $\{c_1, c_2, c_3, c_4\}$ denote the vectors of $\mathcal{C}$. Exhibit the 4×4 matrix C for which column i is $[c_i]_{\mathcal{E}}$.
   (c) Compute the matrices $P = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ and $Q = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.
   [Better alternative: compute a different Q, which is $Q = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}$. Otherwise the transition to the next step is a little mysterious. But you can compute the original Q if you wish.]
   (d) Compute the matrix R such that $R[p]_{\mathcal{B}} = [p]_{\mathcal{C}}$ for every p in $\mathbb{P}_3$.
   (e) What is the $\mathcal{B}$ coordinate vector of the polynomial t?

3. For this problem, we define

$$A = \begin{bmatrix} -3 & -4 & 20 & -8 & -1 \\ 14 & 11 & 46 & 18 & 2 \\ 6 & 4 & -17 & 8 & 1 \\ 11 & 7 & -37 & 18 & 2 \\ 18 & 12 & -60 & 24 & 6 \end{bmatrix} .$$

   (a) Use the `eig` command to find the eigenvalues of A.
   (b) Write p = `det(L*eye(n) - A )` to find the characteristic polynomial of A.
   (c) Use factor(p) to factor this characteristic polynomial.

(Problems continue on the next page.)

4. Consider the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 \\ 3 & 2 & 5 & 0 \\ 4 & 3 & 0 & 3 \end{bmatrix}.$$

(This is a symmetric matrix – equal to its transpose – and we will have a theorem that such a matrix has a basis of eigenvectors.)

(a) Find the eigenvalues of C using `eig`.

(b) Find matrices P, D such that D is diagonal and $P^{-1}CP = D$.

(c) The equation means $CP = PD$, which is a way of writing that the columns of P are eigenvectors. (Moreover, because the columns of P are linearly independent, they form a basis of eigenvectors.) Exhibit CP and PD.

(d) $\star$ MATLAB chose the eigenvectors (columns of P) so that every column would have a certain length. What is it?

(e) Exhibit $P * P'$. What is the relation between the inverse and the transpose of P? Check by exhibiting inv(P). (If we begin with any symmetric matrix C, then Theorem 2 in Section 7.1 of Lay shows we can always find a P of this type.)

5. (a) Exhibit $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

(b) Exhibit [P,D]=eig(A).

(c) $\star$ Do we have $P^{-1}AP = D$?

(d) $\star$ Does A have a basis of eigenvectors? Justify your answer.

6. (a) Exhibit $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(b) Exhibit eig(A).

(c) Exhibit [P,D]=eig(A).

(d) Exhibit inv(P)*A*P .
(If the matrix is diagonalizable but with some nonreal eigenvalues, then MATLAB just goes to work with complex coefficients.)

The last problem is on the next page.

7. A *stochastic* matrix is a square matrix with every entry nonnegative and every row sum equal to one. (In Section 4.9, Lay uses a transposed convention that every column sum is 1, but the row sum definition is the more standard choice.

A stochastic matrix P can be used to define a Markov process: the entry $P(i,j)$ is interpreted as the probability of going from state i to state j, and $P^n(i,j)$ is interpreted as the probability of going from state i to state j in n steps. For example, for P 2×2, state 1 might mean "sunny weather" and state 2 might mean "rainy weather", with $P^n(1,2)$ interpreted as the probability of rainy weather after n days given that today's weather is sunny. Markov models are used a lot. There is more on this in Lay's Section 4.9 .

In this problem we examine the likelihood of moving from one state to another after a delay.

(a) Exhibit the stochastic matrix $P = \begin{pmatrix} .8 & .2 \\ .3 & .7 \end{pmatrix}$.

(b) Exhibit a column vector v with positive entries such that $Av = v$ for *every* 2×2 stochastic matrix A. (This vector v is a right eigenvector of A for the eigenvalue 1.)

(c) Use MATLAB commands to compute a row vector u with positive entries such that $uP = u$ and the entries of u sum to 1. (This vector u is a left eigenvector of P for the eigenvalue 1.)

(You might use [Q,D]=eig(P'), define w to be the transpose of an appropriate column of Q, and then multiply w by a suitably defined scalar.)

(d) Exhibit the matrices $P, P^2, P^5, P^{10}, P^{20}$.

(e) $\star$ Interpreting $P^n(i,j)$ as for a Markov model: you should be seeing in the example that the probability of being in state j after n steps approaches a constant independent of the initial state. What is that constant, in terms of an eigenvector?

(f) Repeat parts (a), (c) and (d) of the problem for the matrix $P = \begin{pmatrix} .6 & 0 & .2 & .2 \\ .1 & .7 & .1 & .1 \\ 0 & .2 & .5 & .3 \\ 0 & .3 & .1 & .6 \end{pmatrix}$.

Here is a remark for your information, in case you are interested.
The behavior above in the powers of a stochastic matrix P is guaranteed, as long as that matrix P has a power for which all entries are positive. Without that condition, the powers of a stochastic matrix P won't necessarily approach a matrix with equal rows (although it might). For examples, you could consider which of the stochastic matrices below have powers converging to a matrix with all rows equal.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$