**Extending Justin's Guide to MATLAB in MATH240 - Part 4**

1. **Method**

   We assume you are comfortable (and remember or can review) commands used in the earlier projects.

2. **New Commands**

   (a) We can compute the dot product, given by the command `dot`. For example:
   `>> dot([1;2;4],[-2;1;5]`

   (b) We can find the length of a vector from the basic definition. If `v` is a vector then:
   `>> sqrt(dot(v,v))`

   (c) Or we can use the `norm` command:
   `>> norm(v)`

   (d) To get the transpose of a matrix `A` we do:
   `>> transpose(A)`
   or
   `>> A'`

   (e) To find the rank of a matrix `A` we do
   `>> rank(A)`

   (f) When $A$ is a matrix with linearly independent columns, the command
   `>> [Q,R]=qr(A,0)`
   will create and exhibit the matrices $Q, R$ which give the $QR$ factorization of $A$ as defined in the text of Lay.

   (g) MATLAB lets you define vectors and submatrices from matrices. For example, given an $m \times n$ matrix $A$, from the definitions
   `>> F=A(1:3,2:4), G = A(1:3, :), H = A(:,2)`
   $F$ is the $3 \times 3$ matrix built from entries of $A$ in rows 1-3 and columns 2-4; $G$ is the $3 \times n$ matrix built out of the first 3 rows; and $H$ is $1 \times n$ matrix (column vector) which equals column 2 of $A$.

3. **Other Commands**

   Just a little review:

   (a) If $A$ is diagonalizable, then the command
   `>> [V,D]=eig(A)`
   produces $D$ diagonal with diagonal entries the eigenvalues of $A$ and $V$ whose columns are the corresponding eigenvectors (so, $AV = VD$, and $V^{-1}AV = D$). Then, for example, the command
   `>> r= V(:,2)`
   will produce as $r$ the corresponding eigenvector.

   (b) A left eigenvector of a square matrix is a row vector $w \neq 0$ such that $wA = \lambda w$, where $\lambda$ is the eigenvalue. To find row eigenvectors you can mix the previous tool with transpose.

   (c) You can also find an eigenvector by solving a linear equation.

   (d) Recall, the default MATLAB format is "short". You can change to format rat or format long as you might find appropriate. The command format short takes you back to the default.

   If you don't know commands to achieve a MATLAB goal, you can go back to earlier projects, or use the help function in MATLAB. You can also guess commands and experiment to see what happens. Just remember that the output you turn in to your T.A. should be clean and devoid of the explorations and mistakes along the way.

4. **Finding Commands**

   If you don't know commands to achieve a MATLAB goal, you can go back to earlier projects, or use the help function in MATLAB. You can also guess commands and experiment to see what happens. Just remember that the output you turn in to your T.A. should be clean and devoid of the explorations and mistakes along the way.

5. **MATLAB Help: an example**

   Suppose $A$ is a $4 \times 3$ matrix with $\text{rank}(A) = 3$. As it turns out, the command

   ```
   >> [Q,R]=qr(A)
   ```

   produces something different from the QR factorization defined as in Lay. What's up?

   In MATLAB, I click on help; there it seems sensible to click on the function browser; then to click mathematics; then linear algebra; then factorization. Scrolling down I click on qr and get the news that I need qr(A,0). (And I see a variety of qr command variants, if I want to know more.)

**MATH 240 Spring 2013 MATLAB Project 4**

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**Directions:**

As before, print out and hand in a record of your final session (the entire contents of the command window). As before, answer any question marked with a star $\star$ by typing in a comment to your output.)

PRODUCE CLEAN, COHERENT OUTPUT FOR YOUR TA. For example, you can copy your successful commands and comments into your favorite editor, and then input them to MATLAB in a final session after you know they work. With comments, make clear what problem you are working on. Also, it is sometimes helpful to add a brief comment explaining the meaning of a computation.

You can collaborate with up to two other students in the course. As before, list your collaborators. We encourage working in such groups. Everyone in a group, though, must type in his/her own session into MATLAB. A T.A. will generally grade a group by grading the output of any member and giving that grade to the whole group.

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1. (Eigenvalues and characteristic polynomial.)

   (a) Exhibit the matrix C=[ 1 2 3 4 0; 0 1 0 1 2; 1 1 0 -1 0; 2 0 1 0 1; 0 0 0 0 2 ] .

   (b) Define a symbolic variable t (t=sym) and compute the characteristic polyonial p of $C$ as a function of t.

   (c) Exhibit the eigenvalues of $C$.

   (d) $\star$ What is the general relation between the roots of the characteristic polynomial, the determinant and the constant term of the characteristic polynomial?

   (e) $\star$ For an $n \times n$ matrix, what is the general relation between the coefficient of $t^{n-1}$ in the characteristic polynomial and the diagonal entries of the matrix?

2. (**Matrix products and dot products**)

   (a) Exhibit the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 3 \\ 2 & 0 & 5 \\ -1 & -2 & -1 \end{pmatrix} .$$

   (b) Exhibit the matrix $A'A$ (here $A'$ is the transpose of $A$).

   (c) $\star$ For a matrix $M$, what in general is the relationship between and entry of $M'M$ and dot products of the rows or columns of $M$ ? State an example computation of an entry of $A'A$ and the corresponding dot product as a doublecheck.

   (d) $\star$ What in general is the relationship between an entry of $MM'$ and dot products of the rows or columns of $M$ ?

3. Define the matrix

$$U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix} .$$

   (a) Enter $U$ into MATLAB.

   (b) Use a short, single-line command to determine if the columns of the matrix $U$ form an orthonormal set.
   [Hint: there is some matrix $V$ such that $U$ is an orthogonal matrix if and only if $VU \ldots$ .]

   (c) $\star$ Very briefly, explain why your computation tells you whether those columns form an orthonormal set, by interpreting entries of $VU$ as dot products.

4. In this problem, use orthogonal projections to find an orthogonal basis $\texttt{v1p}, \texttt{v2p}, \ldots$ for the column space of the matrix

$$C = \begin{pmatrix} 1 & -4 & -2 \\ 2 & 1 & 3 \\ -1 & 0 & 5 \\ 2 & 1 & 0 \end{pmatrix} .$$

   (a) Define vectors `v1` through `v3` equaling the columns. Set `v1p` = v1 .

   (b) Calculate `v2p` using a one-line calculation.
   (Could it be v2p = v2 - dot(v1,v2)/dot(v1,v1) * v1 ?)

   (c) Calculate `v3p` using a one-line calculation.

5. (**Gram-Schmidt made easy**)

   (a) Find the matrices $Q, R$ which give the QR factorization of the matrix $C$ of Problem 4 .

   (b) Use this computation to define a matrix $P$ whose columns are an orthonormal basis for the column space of $C$.

   (c) Check this works by computing $P^{tr}P$.

   (d) $\star$ What is the relationship between the columns of $P$ and the vectors v1,v2,v3 of the last problem?

6. Define

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 6 & 7 \\ -1 & -2 & 7 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$$

   Linear algebra theory tells us that for any $n \times n$ symmetric real matrix, there is a basis of $\mathbb{R}^n$ consisting of eigenvectors; and if two eigenvectors of a symmetric real matrix have distinct eigenvalues, then these eigenvectors are orthogonal.

   (a) Test the theory on $A$. Find its four eigenvalues, and a basis of eigenvectors. Make a computation which shows these eigenvectors are mutually orthogonal.

   (b) Find eigenvectors for the two eigenvalues of $B$, and compute a dot product to show they are not orthogonal.

7. Consider the following example of a square positive matrix with biggest eigenvalue equal to 1. Set $P = \begin{pmatrix} .2 & .8 \\ .8 & .2 \end{pmatrix}$.

   (a) $\star$ Type in the eigenvalues of $P$, from your personal inspection of $P$.

   (b) Check your input with a MATLAB command.

   (c) Find a column vector $r$ such that $Pr = r$.

   (d) Find a row vector $\ell$ such that $\ell P = \ell$.

   (e) Compute $rl$.

   (f) Compute $P^n$ for $n = 1, 2, 5, 10, 20$.

   (g) $\star$ State a conjecture about the limit of $P^n$ as $n$ goes to infinity.

   (h) Exhibit the matrix $Q$ to which the matrices $P^n$ are converging as $n$ goes to infinity.

   (i) $\star$ Briefly explain this convergence using the spectral decomposition of $P$ .

8. (**Least Squares Solutions**)

(a) Continuing with the matrix $C$ above, use a command to check that $\text{rank}(C)$ equals 3 (although from the previous part you know this).
Because $C$ has full rank, there is a unique least-squares solution ("best approximate solution") to the equation

$$Cx = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix} .$$

Let $w$ denote this least squares solution.

(b) Use Theorem 14 of Section 6.5 of Lay to give MATLAB a oneline command to compute $w$.

(c) Use Theorem 15 of Section 6.5 of Lay and your QR computation to give MATLAB a oneline command to compute $w$.

(d) Compute the distance $d$ from $\begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix}$ to the closest point in the column space of $C$.