How matrices multiply finite dimensional volume

1. Notation.
If S is a set in $\mathbb{R}^n$ and A is a matrix, then $A(S)$ denotes the set of all vectors Ax such that x is in S. So, $A(S)$ is the set of outputs you get by applying A to inputs from S.

"vol_n"denotes n-dimensional volume. For example, for $n = 1, 2, 3$ respectively this would be length, area and the usual volume.

Given k vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, the *parallelepiped* generated by $v_1, \ldots, v_k$ is the set of all vectors of the form $c_1v_1 + \cdots + c_kv_k$ such that $0 \le c_i \le 1$, for each i. (If $k = 2$ and the vectors v_1, v_2 are linearly independent, then this is a parallelogram, with corners $0, v_1, v_2$ and $v_1 + v_2$.)

If A is a matrix, then P_A denotes the parallelepiped generated by its column vectors.

2. Finite dimensional volume.
We think of approximating a set S in $\mathbb{R}^n$ by a finite collection of (tiny) n-dimensional cubes which contain it (and which overlap only on their boundaries). Adding up the volumes of the cubes in such a collection, we get an *outer estimate* of the volume of S. Likewise, given a collection of cubes contained in S (and overlapping only on their boundaries), we can add up their volumes to get an *inner estimate* of the volume of S.

If there is a number c such that the inner and outer estimates can be gotten arbitrarily close to c, then we say S has a well defined n-dimensional volume, equal to c.

This definition is analogous to equality of upper and lower Riemann sums in calculus giving a well defined integral. There is a more general (and advanced) definition which gives a bigger collection of sets with well defined volume, but this is enough for us here, and covers the sets you naturally encounter (they have well defined volume).

If the the outer estimates can be made arbitarily small, then we set $\text{vol}_n(S) = 0$.

Below, the subsets S considered are assumed to have well defined volume.

3. How square matrices multiply volume.
Suppose A is an $n \times n$ matrix and S is a subset of $\mathbb{R}^n$. Then

$$\text{vol}_n(A(S)) = |\det(A)| \ \text{vol}_n(S) \ .$$

Note: A changes *every* set S by *the same* multiplicative factor! And we know what it is. For $n = 2$, a proof was outlined in class, and there is a discussion in Lay's Section 3.3. (pp. 180-185). The fundamental ideas of the proof for general n are exactly the same.

One consequence of this is that

$$\text{vol}_n(P_A) = |\det(A)| \ .$$

This is because P_A equals $A(S)$ if S is the unit cube. (The unit cube in $\mathbb{R}^n$ is the parallelepiped spanned by the standard basis vectors $e_1, ..., e_n$. The n dimensional volume of the unit cube is 1, so the n-dimensional volume of P_A is 1 multiplied by $|\det(A)|$.

4. How one-one rectangular matrices multiply volume.

Suppose A is $n \times k$ with $k \le n$, and S is a subset of $\mathbb{R}^k$. Then $A(S)$ is a subset of $\mathbb{R}^n$. Suppose the columns of A are linearly independent. That means that the linear transformation $\mathbb{R}^k \to \mathbb{R}^n$ defined by $x \mapsto Ax$ is one-to-one.

For example, for $k = 2$ and $n = 3$, if S is the unit square in $\mathbb{R}^2$, then $A(S)$ is the parallelogram P_A in $\mathbb{R}^3$, and $\text{vol}_k(P_A)$ is the area of P_A.

We want to know how $\mathrm{vol}_k(A(S))$ relates to $\mathrm{vol}_k(S)$. Let us define the number

$$\delta(A) := \sqrt{|\det(A^{tr}A)|}\ .$$

Here is the bottom line:

$$\mathrm{vol}_k(AS) = \delta(A)\mathrm{vol}_k(S)\ .$$

Once again, multiplication by A changes volume by the same multiplicative factor, for all S. The factor is a little more complicated to compute. (And note: if $k = n$, then $\delta(A) = |\det(A)|$.)

By the way, in the case $k = 2$ and $n = 3$, the number $\delta(A)$ is equal to the norm of the cross product of the columns of A.

5. Examples.

1. If $A = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix}$, then P_A is the parallelogram whose four corners are the column vectors $(0, 0)$, $(2, 0)$, $(5, 3)$ and $(7, 3)$. The area of P_A is $|\det(A)| = |(5)(0) - (2)(3)| = |-6| = 6$.
(Draw the paralleleogram – you can easily check the area is indeed 6.)

2. Let B denote the unit ball in $\mathbb{R}^3$, (the set of points (x_1, x_2, x_3) such that $(x_1)^2 + (x_2)^2 + (x_3)^2 \le 1$. Now define an ellipsoid S and a matrix A as follows:

$$S = \{y : \frac{(y_1)^2}{4} + \frac{(y_2)^2}{9} + (y_3)^2 \le 1\} \qquad \text{and} \qquad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ .$$

Then $y = Ax$ is written out as $(y_1, y_2, y_3) = (2x_1, 3x_2, x_3)$. It follows that x is in B if and only if Ax is a point y in S. That is, $A(B) = S$. Consequently,

$$\mathrm{vol}_3(S) = |\det(A)|\mathrm{vol}_3(B) = 6(4/3)\pi(1)^3 = 8\pi\ .$$

3. The area of the parallelogram P in $\mathbb{R}^3$ generated by the column vectors $(1, 2, 3)$ and $(2, 0, 4)$ is computed from $A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 4 \end{pmatrix}$ by

$$\begin{aligned} \mathrm{area}(P) &= \sqrt{|\det(A^{tr}A)|} \\ &= \sqrt{|\det(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 4 \end{pmatrix})|} \\ &= \sqrt{|\det(\begin{pmatrix} 14 & 14 \\ 14 & 20 \end{pmatrix})|} = \sqrt{84}\ . \end{aligned}$$

6. Making sense of $\mathrm{vol}_k(A(S))$ in $\mathbb{R}^n$.

Continue with A $k \times n$ and with the columns of A being linearly independent (i.e., A has rank k). For simplicity and familiarity, from here I will just write for the case $k = 2$ and $n = 3$, although there is no difference in the general case.

The matrix A, as a linear transformation, maps $\mathbb{R}^2$ one-to-one onto a 2-dimensional subspace of $\mathbb{R}^3$, a plane through the origin. Let us name this plane W. Let $\{u_1, u_2\}$ be an orthonormal basis of W (which you could compute by applying the Gram-Schmidt algorithm to the columns of A). Let U be a 3×2 matrix whose columns are u_1, u_2.

The matrix U^{tr} is a 2×3 matrix. There is a linear transformation $T : W \to \mathbb{R}^2$ defined by $T(x) = U^{tr}x$. If $x \in W$, then there are constants c_1, c_2 such that $x = c_1u_1 + c_2u_2$, and we can compute that $T(x) = (c_1, c_2)$. The map T is an *isometry*: it does not change the length of vectors. For x, y in W we have that $\text{dist}(x, y) = \text{dist}(T(x), T(y))$, since

$$\begin{aligned} \text{dist}(x, y) &= ||x - y|| = ||T(x - y)|| = ||T(x) - T(y)|| \\ &= \text{dist}(T(x), T(y)) \ . \end{aligned}$$

You can think of W as a rigid (infinite) piece of cardboard, and of T as taking that rigid piece of cardboard and laying it on the usual plane $\mathbb{R}^2$. For example, T will take a triangle with corners x, y, z in S to a triangle with corners $T(x), T(y), T(z)$ in $\mathbb{R}^2$. The corresponding sides of the two triangles will have equal length. The two triangles are congruent.

However we go about defining area of subsets of W, in the end the area of a set in W must equal the area of its image in $\mathbb{R}^2$ under T.

Now suppose S is a subset of $\mathbb{R}^2$. The area of its image $A(S)$ in W will be the area of $U^{tr}(A(S))$ in $\mathbb{R}^2$. But this is just the image of S under multiplication by the 2×2 matrix $U^{tr}A$. So,

$$\text{area}(A(S)) = |\det(U^{tr}A)|\text{area}(S) \ .$$

Given A, we could compute U and then compute the area-multiplier number $|\det(U^{tr}A)|$. In this sense, we have already answered the question of how multiplication by A changes area. But we can get a much more practical computation by showing that $|\det(U^{tr}A)|$ equals $\delta(A)$, which can be computed directly without bringing in a computation of U.

7. Showing that $\delta(A)$ is the area multiplier.

Let

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

denote the 2×2 matrix which is $U^{tr}A$.

We have our orthonormal basis of column vectors u_1, u_2 for W. Let u_3 be another unit vector, orthogonal to u_1 and u_2. Let u_{ij} denote entry j of u_i. Define the matrix

$$V = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \ .$$

The matrix V is an orthogonal matrix: V is invertible and $V^{-1} = V^{tr}$. Now compute the matrix product

$$\begin{aligned} VA &= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M \\ 0 \end{pmatrix} \ . \end{aligned}$$

The bottom row of VA is zero because u_3 is orthogonal to every vector in W, in particular to the columns of A. The other rows are the rows of M because the first two rows of V are those of U^{tr}. In the last equality we are just writing out the matrix as a block (partitioned) matrix; the zero below the M is the bottom row, with two numerical entries equal to zero.

As a consequence, then,

$$\begin{aligned} (VA)^{tr}VA &= \begin{pmatrix} M \\ 0 \end{pmatrix}^{tr} \begin{pmatrix} M \\ 0 \end{pmatrix} = \begin{pmatrix} M^{tr} 0^{tr} \end{pmatrix} \begin{pmatrix} M \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} M^{tr}M \end{pmatrix} \ . \end{aligned}$$

We have a simple computation:

$$(VA)^{tr}VA = A^{tr}V^{tr}VA = A^{tr}IA = A^{tr}A$$

and therefore

$$\det(A^{tr}A) = \det(M^{tr}M) = \det(M)^2$$

so $\delta(A) = |\det(M)|$ as required.

7. One more example. Let us compute the area of the parallelgram P in $\mathbb{R}^3$ generated by the column vectors $(2, 5, 0)$ and $(3, 1, 0)$. Define matrices

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} .$$

Our formula for the area of P should agree with $\text{area}(P) = |\det(B)|$. Let's check.

By the nature of matrix multiplication,

$$\begin{aligned} A^{tr}A &= \begin{pmatrix} 2 & 5 & 0 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \\ &= B^{tr}B \ . \end{aligned}$$

Therefore

$$\text{area}(P) = \sqrt{\det(A^{tr}A)} = \sqrt{\det(B^{tr}B)} = |\det(B)|$$

as desired.