THE CHEBYSCHEV INEQUALITY

Suppose X is a random variable with a well defined expected value, $E(X) = \mu$, and a well defined, positive variance, i.e.

$$\mathrm{E}((X-\mu)^2) = \sigma^2 > 0.$$

The Chebyschev inequality says that in this case, for any positive number k,

$$\mathrm{Prob}(|X-\mu| \geq k\sigma) \ \leq \ \frac{1}{k^2} \ .$$

For large k, the Chebyshev inequality is saying, in a quantitative way, that on the scale of σ, the probability is small that a random variable will have large outputs. The inequality supports again the view that a random variable is often best understood on the scale of σ.

We will prove the inequality in the special case that X is a discrete random variable.

CASE I: $\mu = 0$.

Here $\sigma^2 = \mathrm{E}(X^2) - \mu^2 = \mathrm{E}(X^2)$. Let $p(x)$ denote the probability that $X = x$. Then we have

$$\begin{aligned}
\sigma^2 = E(X^2) &= \sum_x x^2 p(x) && \text{here add over all } x \\
&\geq \sum_{|x|\geq k\sigma} x^2 p(x) && \text{here add only terms with } |x| \geq k\sigma \\
&\geq \sum_{|x|\geq k\sigma} (k\sigma)^2 p(x) && \text{since } x^2 \geq (k\sigma)^2 \text{ for such terms} \\
&\geq k^2\sigma^2 \sum_{|x|\geq k\sigma} p(x) \\
&= k^2\sigma^2 \mathrm{Prob}(|X| \geq k\sigma).
\end{aligned}$$

Take the beginning and end of this string of inequalities, and divide by $k^2\sigma^2$ to get

$$\frac{1}{k^2} \geq \mathrm{Prob}(|X| \geq k\sigma) \ .$$

This finishes the proof.

CASE II: $\mathrm{E}(X)$ is not assumed to be zero.

Let $Y = X - \mu$. Then $\mathrm{E}(Y) = \mathrm{E}(X) - \mu = 0$. Also, X and Y have the same standard deviation. So, by Case I, we have for any $k > 0$ that

$$\mathrm{Prob}(|Y| \geq k\sigma) \;\leq\; \frac{1}{k^2} \,.$$

Just substitute $X - \mu$ for Y. We're done.

(When X has a continuous distribution given by a density function, the same ideas apply, by change of the sums to integrals.)