PROOF OF THE LAW OF LARGE NUMBERS
IN THE CASE OF FINITE VARIANCE

THEOREM (The Law of Large Numbers)
Suppose $X_1, X_2, \ldots$ are i.i.d. random variables, each with expected value μ.
Then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \text{Prob}\left[\ \left|\frac{X_1+X_2+\cdots+X_n}{n} - \mu\right| \ > \epsilon\ \right] \quad = \quad 0\ .$$

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We will prove the LLN in the special case that the i.i.d. random variables X_i have finite variance σ^2. If $\sigma^2 = 0$, then there is some number b such that each $X_i = b$ with probability one (and the conclusion of the LLN is pretty obvious!). So without loss of generality we suppose $\sigma^2 > 0$. Because the X_i are independent, we know that

$$\text{Var}(X_1+X_2+\cdots+X_n) = \text{Var}(X_1)+\text{Var}(X_2)+\cdots+\text{Var}(X_n)\ .$$

Therefore

$$\text{Var}(X_1+X_2+\cdots+X_n) = n\sigma^2\ .$$

Multiplying a random variable by $1/n$ changes the variance by a factor $(1/n)^2$, so

$$\text{Var}\Big(\frac{X_1+X_2+\cdots+X_n}{n}\Big) = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}\ .$$

Let A_n denote the average $(X_1+X_2+\cdots+X_n)/n$. So, A_n has standard deviation $\sqrt{\sigma^2/n} = \sigma/\sqrt{n}$, and A_n has expected value μ. Given $\epsilon > 0$,

$$\text{Prob}\,[\ |A_n - \mu|\ > \epsilon\ ] = \text{Prob}\left[\ |A_n-\mu|\ > (\epsilon\frac{\sqrt{n}}{\sigma})(\frac{\sigma}{\sqrt{n}})\ \right]\ .$$

Let $k = \epsilon\sqrt{n}/\sigma$. According to the Chebyschev Inequality, since A_n has standard deviation $\sigma/\sqrt{n}$, we have

$$\text{Prob}(|A_n-\mu| \geq k\frac{\sigma}{\sqrt{n}}) \ \leq\ \frac{1}{k^2}\ .$$

Because $k = \epsilon\sqrt{n}/\sigma$, the right hand side $1/k^2$ goes to zero as n goes to infinity. That gives the proof.