THE NORMAL DISTRIBUTIONS $\mathcal{N}(\mu, \sigma)$

Let μ and σ be real numbers, with $\sigma > 0$, and suppose X is a random variable with mean μ and standard deviation σ. To say that X has the $\mathcal{N}(\mu, \sigma)$ distribution is the same thing as saying that its standardized version

$$Z = \frac{X - \mu}{\sigma}$$

has the standard normal distribution $\mathcal{N}(0, 1)$, that is, Z has the distribution given by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \ .$$

Note, we have $X = \mu + \sigma Z$, and for any numbers a, b with $a \le b$, each of the following conditions is equivalent:

$$\begin{array}{rcccl}
a & \le & Z & \le & b \ , \\
a\sigma & \le & \sigma Z & \le & b\sigma \ , \\
\mu + a\sigma & \le & \mu + \sigma Z & \le & \mu + b\sigma \ , \\
\mu + a\sigma & \le & X & \le & \mu + b\sigma \ , \\
a\sigma & \le & X - \mu & \le & b\sigma \ .
\end{array}$$

Because these conditions define the same event, they have the same probability, in particular

$$\begin{aligned}
\mathrm{Prob}(a \le Z \le b) &= \mathrm{Prob}(a\sigma \le X - \mu \le b\sigma) \\
&= \mathrm{Prob}(\mu + a\sigma \le X \le \mu + b\sigma) \ .
\end{aligned}$$

For example, if $\mu = 7$ and $\sigma = 3$, and we take $a = -2, b = 2$, then

$$\begin{aligned}
\mathrm{Prob}(-2 \le Z \le 2) &= \mathrm{Prob}(-2(3) \le X - 7 \le 2(3)) \\
&= \mathrm{Prob}(7 - 2(3) \le X \le 7 + 2(3)) \ .
\end{aligned}$$

SO: the distribution $\mathcal{N}(\mu, \sigma)$ looks just like $\mathcal{N}(0, 1)$, except it is recentered at μ and rescaled by σ.