Polynomial approximation and higher derivatives

Suppose $f : I \to \mathbb{R}$, where I is an open interval and $x_0 \in I$, and suppose f is differentiable at x_0. Then we can easily show

$$\lim_{x\to x_0} \frac{f(x) \;-\; (\,f(x_0)+f'(x_0)(x-x_0)\,)}{x-x_0} \;=\; 0\ .$$

Because $p_1(x) = f(x_0) + f'(x_0)(x - x_0)$ is the first Taylor polynomial of f at x_0, we see a natural question, which is answered by the following result.

Theorem 0.1. *Suppose n is a positive integer, I is an open interval containing x_0, $f : I \to \mathbb{R}$, $f^{(n-1)}$ exists on I and $f^{(n)}(x_0)$ exists. Let p_n be the nth Taylor polynomial of f at x_0. Then*

$$\lim_{x\to x_0} \frac{f(x) \;-\; p_n(x)}{(x-x_0)^n} \;=\; 0\ .$$

Proof. For notational simplicity, we will verify just the onesided limit as $x \to x_0^+$. A very similar argument shows the limit from the left is also zero.

We prove the theorem by induction on n. For $n = 1$, we know the theorem is true. Now suppose it is true for a positive integer $n - 1$. We will show the theorem is then true for n.

Without loss of generality (after replacing f with $f - p_n$) we may assume $f^{(k)}(x_0) = 0$ for $0 \le k \le n$. It then holds that $f - p_n = f$. Also without loss of generality, we assume for simplicity that $x_0 = 0$.

Now suppose $\epsilon > 0$. Applying the induction hypothesis to f', we choose $\delta > 0$ such that $0 < |x| < \delta$ implies $|f'(x)| < \epsilon|x|^{n-1}$. Suppose $0 < x < \delta$. Given a positive integer M, split the interval $[0, x]$ into M subintervals of equal length x/M. The ith subinterval $[x_{i-1}, x_i]$ is $[(i-1)x/M, ix/M]$. By the Mean Value theorem, in each subinterval $[x_{i-1}, x_i]$ there is a point x_i^* such that

$$|f(x_i) - f(x_{i-1})| = |f'(x_i^*)|(x/M)\ .$$

Because $|f'(x_i^*)| \;\le\; \epsilon(x_i^*)^{n-1} \le \epsilon(ix/M)^{n-1}$ and

$$\begin{aligned} f(x) \;&=\; f(x)-f(0) \;=\; f(x_M)-f(0) \\ &=\; \Big(f(x_M)-f(x_{M-1}\Big)+\Big(f(x_{M-1})-f(x_{M-2}\Big)+\cdots+\Big(f(x_1)-f(x_0)\Big) \\ &=\; \sum_{i=1}^{M}(f(x_i)-f(x_{i-1})) \end{aligned}$$

we have

$$\begin{aligned} |f(x)| \;&=\; |\sum_{i=1}^{M}(f(x_i)-f(x_{i-1}))| \;\le\; \sum_{i=1}^{M}|(f(x_i)-f(x_{i-1})| \\ &\le\; \sum_{i=1}^{M}\Big(\epsilon(ix/M)^{n-1}\Big)(x/M)\ . \end{aligned}$$

We recognize this sum as a Riemann sum for the integral $\int_{t=0}^{x} \epsilon t^{n-1}\, dt$ using our regular partition $\{x_i : 0 \le i \le M\}$. Because $\int_{t=0}^{x} \epsilon t^{n-1}\, dt = \epsilon x^n/n$, we have

$$\begin{aligned} |f(x)| &\le \lim_{M\to\infty} \sum_{i=1}^{M} \Big(\epsilon (ix/M)^{n-1}\Big)(x/M) \\ |f(x)| &\le \epsilon x^n/n \ . \end{aligned}$$

Therefore, $0 < x < \delta$ implies $|f(x)|/x^n \le \epsilon/n$. Because ϵ was arbitary, this proves the theorem. □

REMARK. There is an easier proof of the theorem under an additional hypothesis. A function is called C^k on an interval if its kth derivative is well defined and continuous on that interval.

Easier Proof assuming f is C^n on I.

As before we may assume $p_n = 0$ and $x_0 = 0$. By the Lagrange Remainder Theorem, given $x \in I$ and $x \ne 0$, we have a z between 0 and x such that

$$\begin{aligned} f(x) - p_{n-1}(x) &= \frac{f^{(n)}(z)}{n!} x^n \ , \quad \text{and therefore} \\ \frac{f(x) - p_{n-1}(x)}{x^n} &= \frac{f^{(n)}(z)}{n!} \ . \end{aligned}$$

Because $p_n = 0 = p_{n-1}$, we can replace p_{n-1} above with p_n. Also, by assumption of continuity of $f^{(n)}$, and because $z \to 0$ as $x \to 0$, we then have

$$\lim_{x\to 0} \frac{f(x) - p_n(x)}{x^n} = \lim_{z\to 0} \frac{f^{(n)}(z)}{n!} = 0 \ .$$