Continuity and uniform continuity with epsilon and delta

We will solve two problems which give examples of working with the ϵ, δ definitions of continuity and uniform continuity.

Problem. Show that the square root function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Solution. Suppose $x \geq 0$ and $\epsilon > 0$. It suffices to show that there exists a $\delta > 0$ such that for every y in the domain (i.e. every $y \geq 0$)

$$|x - y| < \delta \implies |\sqrt{x} - \sqrt{y}| < \epsilon \ .$$

Note, unless both x and y are zero, we have

$$\begin{aligned} \sqrt{x} - \sqrt{y} &= (\sqrt{x} - \sqrt{y})\left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) \\ &= \frac{x - y}{\sqrt{x} + \sqrt{y}} \ . \end{aligned}$$

Case I: $x > 0$. Choose $\delta = x\sqrt{\epsilon}$. Then

$$|x-y| < \delta \implies |\sqrt{x}-\sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon \ .$$

Case II: $x = 0$. Choose $\delta = \epsilon^2$. It suffices to consider $y > 0$, and then

$$|x-y| < \delta \implies |\sqrt{x}-\sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} = \frac{y}{\sqrt{y}} = \sqrt{y} \ < \sqrt{\epsilon^2} = \epsilon \ .$$

This finishes the proof that the square root function is continuous.

Problem. Show that f is uniformly continuous.

A solution. Suppose $\epsilon > 0$. By the epsilon-delta definition of uniform continuity, it suffices to show that there exists $\delta > 0$ such that for all x, y in the domain of f,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon \ .$$

Because f is continuous on the closed bounded interval $[0,2]$, f is uniformly contiuous on $[0,2]$. Thus there exists $\delta_1 > 0$ such that whenever x and y are both in $[0,2]$, we have

$$|x-y| < \delta_1 \implies |f(x) - |f(y)| < \epsilon \ .$$

Next, if x and y are both at least 1, we have

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2} \ .$$

Choose $\delta = \min\{\delta_1, 1, \epsilon\}$ and suppose $|x-y| < \delta$. If one of x, y is less than 1, then both are smaller than 2 (since $\delta \leq 1$) and then $|f(x) - f(y)| < \epsilon$ (since $\delta \leq \delta_1$). On the other hand, if x and y are both in $[1, \infty)$, then $|f(x) - f(y)| < \frac{|x-y|}{2} < \frac{\epsilon}{2} < \epsilon$. This proves that f is uniformly continuous.

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There are other ways to do the exercises above. One useful trick to remember is the multiplication move with the square roots. Another technique in the example, which is used over and over and over, is to split a problem into more than one case, handle the cases separately, and then put the solutions together to solve the original problem.