JOINTLY PERIODIC POINTS IN CELLULAR AUTOMATA: COMPUTER EXPLORATIONS AND CONJECTURES

MIKE BOYLE AND BRYANT LEE

Abstract. We develop a rather elaborate computer program to investigate the jointly periodic points of one-dimensional cellular automata. The experimental results and mathematical context lead to questions, conjectures and a contextual theorem.

Contents

1. Introduction and conjectures

In this paper we consider the action of a surjective one-dimensional cellular automaton f on jointly periodic points. Detailed definitions are recalled below.

This paper is primarily an experimental mathematics paper, based on data from a program written by the second-named author to explore such actions. The experimental results and mathematical context lead us to questions and a conjecture on the growth rate of the jointly periodic points.

We approach our topic from the perspective of symbolic dynamics, which provides some relevant tools and results. However, almost all of this paper–in particular the questions and conjectures–can be well understood without symbolic dynamics. We do spend time on context, and even prove a theorem (Theorem 3.2), for two reasons. First, we believe that experimental mathematics should not be too segregated from the motivating and constraining mathematics. Second, workers on cellular automata have diverse backgrounds, not necessarily including symbolic dynamics. (Similarly, perhaps a technique or example unfamiliar to us could resolve one of our questions.)

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To express our questions and conjectures clearly, we must suffer some definitions. We let Σ_N denote the set of doubly infinite sequences $x = \dots x_{-1}x_0x_1\dots$ such that each x_i lies in a finite "alphabet" $\mathcal{A}$ of N symbols; usually $\mathcal{A} = \{0, 1, \dots, N-1\}$. A one-dimensional cellular automaton (c.a.) is a function $f : \Sigma_N \to \Sigma_N$ for which there are integers $a \leq b$ and a function $F : \mathcal{A}^{b-a+1} \to \mathcal{A}$ (F is a "local rule" for f) such that for all i, $(f(x))_i = F(x_{i+a} \cdots x_{i+b})$. The shift map σ on a sequence is defined by $(\sigma x)_i = x_{i+1}$. We let S_N denote the shift map on Σ_N.

For any map S, we let $P_k(S)$ denote the points of (not necessarily least) period k of S, i.e. the points fixed by S^k, and let $\mathrm{Per}(S) = \cup_k P_k(S)$. Thus, $\mathrm{Per}(S_N)$ is the set of "spatially periodic" points for a one-dimensional cellular automaton on N symbols. The *jointly periodic* points of a cellular automaton map f on N symbols are the points in $\mathrm{Per}(S)$ which are also periodic under f, that is, the points which are "temporally periodic" as well as spatially periodic. (In the usual computer screen display, this would mean vertically and well as horizontally periodic.) There is by this time a lot of work addressing periodic and jointly periodic points for *linear* one-dimensional c.a.; we refer to [Chin-Cortzen-Goldman2001, Cordovil-Dilao-daCosta1986, Jen1988, Lidman-Thomas2006, Martin-Odlyzko-Wolfram1984, Misiurewicz-Stevens-Thomas2006, Sutner] and their references. Also see [Miles2006] regarding the structure of periodic points for these and more general algebraic maps in the setting of [Kitchens1997, Schmidt1995].

A subset E of Σ_N is *dense* if for every point x in Σ_N and every $k \in \mathbb{N}$ there exists y in E such that $x_i = y_i$ whenever $|i| \leq k$. We say E is m-dense if every word of length m on symbols from the alphabet occurs in a point of E.

We can now state our first conjecture.

Conjecture 1.1. *For every surjective one-dimensional cellular automaton, the jointly periodic points are dense.*

Conjecture 1.1 is a known open question [Blanchard2000, Blanchard-Tisseur2000, Boyle-Kitchens1999], justified by its clear relevance to a dynamical systems approach to cellular automata. (Whether points which are temporally but not necessarily spatially periodic for a surjective c.a. must be dense is likewise unknown [Blanchard2000].) That this question, also open for higher dimensional c.a., has not been answered reflects the difficulty of saying anything of a general nature about c.a., for which meaningful questions are often undecidable [Kari2005].

It is known that the jointly periodic points of a one-dimensional cellular automaton map f are dense if f is a closing map [Boyle-Kitchens1999] or if f is surjective with a point of equicontinuity [Blanchard-Tisseur2000]. We justify our escalation of (1.1) from question to conjecture by augmenting earlier results with some experimental evidence. In particular: for every span 4 surjective one-dimensional cellular automaton on two symbols, the jointly periodic points are at least 13-dense (Proposition 5.1).

Now we turn to more quantitative questions. Letting for the moment P denote the number of points in $P_k(S_N)$ which are periodic under f as well as S_N (i.e. $P = |\mathrm{Per}(f|P_k(S_N))|$), we set $\nu_k(f, S_N) = P^{1/k}$, and then define

$$\nu(f, S_N) = \limsup_k \nu_k(f, S_N) \ .$$

Question 1.2. *Is it true for every surjective one dimensional cellular automaton f on N symbols that $\nu(f, S_N) \geq \sqrt{N}$?*

Question 1.3. *Is it true for every surjective one dimensional cellular automaton f on N symbols that $\nu(f, S_N) > 1$?*

We cannot answer Question 1.3 even in the case that f is a "closing" map and we know there is an abundance of jointly periodic points [Boyle-Kitchens1999].

Conjecture 1.4. *There exists $N > 1$ and a surjective cellular automaton f on N symbols such that $\nu(f, S_N) < N$.*

Conjecture 1.4 is a proclamation of ignorance. ¿From the experimental data in our Tables, it seems perfectly clear that there will be many surjective c.a. f with $\nu(f, S_N) < N$. However, we are unable to give a proof for any example. With the additional assumption that the c.a. is linear, it is known that Conjecture 1.1 is true and the answer to Question 1.2 is yes (Sec. 3).

The relation of Questions (1.2-1.4) to Conjecture 1.1 is the following: if a c.a. map f on N symbols does not have dense periodic points, then $\nu(f, S_N) < N$.

Here is the organization of the sequel. In Section 2, we give detailed definitions and background. In Section 3, we establish some mechanisms by which one can prove lower bounds for $\nu(f, S_N)$ for some f. We also prove (Theorem 3.2) that no property of a surjective c.a. considered abstractly as a quotient map without iteration can establish $\nu(f, S_N) < N$. We also support Question 1.2 with a random maps heuristic. (The potential analogy of c.a. and random maps was remarked earlier by Martin, Odlyzko and Wolfram [Martin-Odlyzko-Wolfram1984, p.252] in their study of linear c.a.) A list of c.a. used for the computer explorations is given in Section 4.

Our computer program consists of three related subprograms: FDense, FPeriod and FProbPeriod. We use these respectively in Sections 5 , 6 and 7. FDense probes approximate density of jointly periodic points of a given shift period. FPeriod provides exact information on jointly periodic points of a given shift period. FProbPeriod provides information on jointly periodic points for a random sample from a given shift period, and thus provides some information at shift periods where the memory demands of FPeriod are too great for it to succeed.

In Sections 5, 6 and 7, we give more information on the algorithms and discuss the many tables of output data in the appendices. The tables, along with the program itself, are available as an online supplement at the Experimental Mathematics website (http://www.expmath.org/expmath/volumes/VOLX/VOLX.ISSX/), and also at the website of the first named author.

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2. Definitions and background

Let $\mathcal{A} = \{0, 1, \dots, N-1\}$, a finite set of N symbols, with the discrete topology. Let Σ_N be the product space $\mathcal{A}^{\mathbb{Z}}$, with the product topology. We view a point x in Σ_N as a doubly infinite sequence of symbols from $\mathcal{A}$, $x = \dots x_{-1}x_0x_1 \dots$. The space Σ_N is compact and metrizable; one metric compatible with the topology is $\text{dist}(x, y) = 1/(|n| + 1)$ where $|n|$ is the minimum nonnegative integer such that $x_n \neq y_n$. A set E is *dense* in Σ_N in this topology if for every k and every word W in $\mathcal{A}^{2k+1}$ there exists x in E such that $x[-k, k] = W$.

The shift map σ sends a sequence x to the sequence σx defined by $(\sigma x)_i = x_{i+1}$. The shift map defines a homeomorphism S_N on Σ_N. The topological dynamical system (Σ_N, S_N) is called the *full shift on N symbols*, or more briefly the N-shift.

A map $f : \Sigma_N \to \Sigma_N$ is continuous and shift-commuting ($f\sigma = \sigma f$) if and only f is a *block code*, i.e. there exist integers a, b and a function $F : \mathcal{A}^{b-a+1} \to \mathcal{A}$ such that $(f(x))_i = F(x[i+a, i+b])$ for integers i, for all $x \in \Sigma_N$. Such a map f is called a one-dimensional cellular automaton. There is a well known dichotomy for such maps f: either (i) f is surjective and for some integer M every point has at most M preimages, or (ii) image points typically have uncountably many preimages, and f is not surjective [Hedlund1969, Kitchens1998, Lind-Marcus1995]. In Case (i), almost all points have the same number of preimages; this number is the *degree* of f.

We restrict our attention to surjective maps in this paper because we are interested in periodic points of f, which must be contained in $\cap_{k>0} f^k \Sigma_N$, the eventual image of f. We separate our ignorance about periodic points from additional difficulties involving the passage to the eventual image [Maass1995].

Polynomials can be used to define cellular automata; for example, if we refer to the c.a. f defined on the N-shift by the polynomial $2x_{-1} + x_0(x_2)^3$, we mean that f is defined by the block code $(fx)_i = 2x_{i-1} + x_i(x_{i+2})^3$, where the arithmetic is interpreted modulo N. The *span* of such a code is 1 plus the maximum difference of coordinates with nonzero coefficients; in this example, it is $1 + 2 - (-1) = 4$. The code is left permutative if for every x, permuting inputs to the leftmost variable, with inputs to other variables fixed, permutes the outputs. Likewise there is the notion of right permutative. The previous example is left permutative and it is not right permutative. When the number N of symbols is prime, every c.a. map f has such a polynomial representation [Hedlund1969]. (For general N, there is a representation by a product of polynomial representations over finite fields [Martin-Odlyzko-Wolfram1984].)

A block code on S_N depending on coordinates $[0, j-1]$ can be described by a "lookup code", a word W of length N^j on alphabet $\{0, ..., j-1\}$ defined as follows. List the N^j possible blocks of length j in lexicographic order; then the ith symbol of W is the output symbol under f for the ith input block. For example, for the code $x_0 + x_1 x_2$ on S_2, the input words in lexicographic order are 000,001,010,011, 100,101,110,111 and the corresponding word W is 0001 1110.

For the N shift, the number of coding rules of span at most j is N^{N^j}. If $\mathrm{inj}(j, N)$ denotes the number of these which define injective (and thus surjective [Hedlund1969]) codes, then we still [Kim-Roush1990] see a superexponential growth rate in j,

$$\lim_j \frac{1}{j} \log\log(\mathrm{inj}(j, N)) = \log N$$

even though surjective span j maps become very sparse in the set of all span j maps as j increases.

A block code $f : \Sigma_N \to \Sigma_N$ is *right-closing* if it never collapses distinct left-asymptotic points. This means that if $f(x) = f(x')$ and for some I it holds that $x_i = x'_i$ for all i in $(-\infty, I]$, then $x = x'$. Any right permutative map is right closing. The definition of *left closing* is given by replacing $(-\infty, I]$ with $[I, \infty)$. The map f is *closing* if it is either left or right closing. An endomorphism of a full shift S_N is constant-to-one if and only if it is both right and left closing (i.e., it is biclosing). A closing map is surjective. Closing maps are important in the coding theory of symbolic dynamics [Ashley1993, Kitchens1998, Lind-Marcus1995]. They also have a very natural description from the viewpoint of hyperbolic dynamics [Brin-Stuck2002]: right closing maps are injective on unstable sets, left closing maps are injective on stable sets.

We now discuss some previous work involving periodic points and cellular automata. We let $P_n(S)$ denote the points of period n of S, and $P_n^o(S)$ the points of least period n. These finite sets are mapped into themselves by any c.a. map f; thus any periodic point of S is at least preperiodic for f. For a preperiodic (possibly periodic) point x, the *preperiod* of x is the least nonnegative integer j such that $f^j(x)$ is periodic, and the *period* of x is its eventual period, the smallest positive integer k such that $f^{m+k}(x) = f^m(x)$ for all large m. A point is jointly periodic if it is periodic under both f and S_N.

In the case f is linear $(f(x) + f(y) = f(x + y))$, Martin, Odlyzko and Wolfram [Martin-Odlyzko-Wolfram1984] (see also the further work in [Chin-Cortzen-Goldman2001, Cordovil-Dilao-daCosta1986, Jen1988, Lidman-Thomas2006, Misiurewicz-Stevens-Thomas2006, Sutner] and their references) gave an algebraic analysis of f-periods and preperiods for points of a given shift period, and also provided some numerical data. One key feature for linear f is an easy observation: among the jointly periodic points of shift period k, there will be a point (generally many points) whose least f-period will be an integer multiple of all the least f-periods of the jointly periodic points of shift period k. In contrast, a very special case of a powerful theorem of Ashley [Ashley1993] has the following statement: for any K, N and any shift-commuting map g from $\cup_{1\le k\le K} P_k(S_N)$ to itself, there will exist surjective c.a. on N symbols whose restriction to $\cup_{1\le k\le K} P_k(S_N)$ equals g.

The following remark is another indication of the difficulty of understanding joint periodicity of even injective c.a. For a map T, $\mathrm{Fix}(T)$ denotes $P_1(T)$, the set of fixed points of T.

Remark 2.1. Given $N \ge 2$, let S denote S_N, and suppose ϕ is an injective one-dimensional c.a. on N symbols. Suppose N is prime. Then there will exist some integer m, depending on ϕ, and some $\kappa > 0$ such that for all $k \in \mathbb{N}$,

$$\Big|\mathrm{Fix}\big((S^a\phi^b)^k\big)\Big| = N^{(a+mb)k} = \Big|\mathrm{Fix}\big((S^a(S^m)^b)^k\big)\Big| = \Big|\mathrm{Fix}\big((S^{a+mb})^k\big)\Big|$$

whenever $|b/a| < \kappa$ (this follows from [Boyle-Krieger1987, Theorem 2.17]). That is, for the two $\mathbb{Z}^2$ actions generated respectively by S, ϕ and S, S^m, the periodic point counts for actions by individual elements (a, b) of $\mathbb{Z}^2$ are the same for all (a, b) in some open cone around the positive horizontal axis. However, despite the agreement in that open cone, the sequences $(|P_k(\phi)|)$ and $(|P_k(S^m)|)$ can be very different. (For a dramatic example of this sort in the setting of shifts of finite type, see [Nasu1995, Example 10.1]).

Lastly, we note that the invariant ν, defined in the introduction, has an unusual robustness, as follows.

Remark 2.2. Fix N and let $S = S_N$. Suppose $x \in \mathrm{Per}(S_N)$ and f is a c.a. on Σ_N. Then x is in $\mathrm{Per}(f)$ if and only if for some $i > 0$, $f^i x$ and x are in the same S-orbit. It follows that for all integers i, j, k with k, i positive and j nonnegative, we have $\nu_k(f, S) = \nu_k(f^i S^j, S)$, and thus $\nu(f, S) = \nu(f^i S^j, S)$.

3. Some mechanisms for periodicity

Throughout this section f denotes a c.a. map on N symbols. In this section, we discuss four ways to prove $\nu(f, S_N)$ is large:

(1) find a large shift fixed by f (or more generally by a power of f)
(2) let f be linear (i.e., a group endomorphism of Σ_N, where addition on the compact group Σ_N is defined coordinatewise mod N)
(3) use the algebra of a polynomial presenting f
(4) find equicontinuity points.

After discussing these, we offer a random maps heuristic and a question.

(1) We will exhibit the first mechanism in some generality. Two c.a. f, g are isomorphic if there is an invertible c.a. ϕ such that $f = \phi g \phi^{-1}$ (where e.g. ϕg is the composition, $(\phi g)(x) = \phi(g(x))$). The c.a. f, g are equivalent as quotient maps if there are invertible c.a. ϕ, ψ such that $f = \psi g \phi$. We prove Theorem 3.2 below to show that for a c.a. f, no property defined on equivalence classes of quotient maps can prevent $\nu(f, S_N)$ from being arbitrarily close to N. To avoid a lengthy digression to background, we give a proof assuming some familiarity with symbolic dynamics; however the statement of Theorem 3.2 is self-contained. Below, $h(T)$ denotes the topological entropy of T.

Lemma 3.1. *Suppose (Σ, S) is a mixing shift of finite type (SFT) of positive entropy, and $f : \Sigma \to \Sigma$ is a surjective block code, and $\delta > 0$. Then there are an automorphism ϕ of (Σ, S), and a mixing SFT (Σ', S') such that the following hold: $\Sigma' \subset \Sigma$; the fixed point set of ϕf contains Σ'; and $h(S') > h(S) - \delta$.*

Proof. In this proof, we will consider only SFTs which are restrictions of S to subsets of Σ. For a lighter notation, we will let the set name also denote the SFT which is the restriction of S to the set.

Our first task is to find a mixing SFT X in Σ such that $h(X) > h(\Sigma) - \delta$ and the restriction of f to X is injective. For this, pick a periodic orbit Z in Σ such that $f(Z)$ is an orbit of equal period. (Such Z must exist: otherwise, f would map the periodic points of prime period to fixed points, and this would imply that $f(\Sigma)$ is a single point.) As a surjective endomorphism of a mixing SFT, the map f must be finite to one, so $f^{-1}(f(Z))$ is a finite set. Using [Denker-Grillenberger-Sigmund1976, Lemma 26.17]), find in Σ a mixing SFT Σ_1 disjoint from the subshift $f^{-1}(f(Z)) \setminus Z$, such that $h(\Sigma_1) > h(\Sigma) - \delta$. Then, using [Denker-Grillenberger-Sigmund1976, Lemma 26.16]), find in Σ a mixing SFT Σ_2 containing $Z \cup \Sigma_1$ but disjoint from $f^{-1}(f(Z)) \setminus Z$. Now $h(\Sigma_2) > h(\Sigma) - \delta$ and the finite to one map $f|\Sigma_2$ is injective on Z. Thus the restriction of f to Σ_2 has degree 1. Let W be a magic word for this degree 1 map (see [Kitchens1998] or [Lind-Marcus1995, Sec. 9.1] for background on magic words and degree). Define

$$Y_M = \{x \in f(\Sigma_2) : \forall i \in \mathbb{Z}, W \text{ is a subword of } x[i, i+M]\} .$$

Any point of Y_M will have a unique preimage in Σ_2. As in [Marcus1985], $\lim h(Y_M) = h(f(\Sigma_2))$. Fix M sufficiently large that Y_M is a mixing SFT with entropy close enough to that of $f(\Sigma_2)$ to guarantee $h(Y_M) > h(\Sigma) - \delta$. Let X denote $\Sigma_2 \cap f^{-1}(Y_M)$. Then X is a mixing SFT with $h(X) > h(\Sigma) - \delta$ and $f|X$ is injective.

Now fix K such that for every $n \geq K$, Σ has at least two orbits of length n which are not in X. Find a mixing SFT Σ' in X such that Σ' (and consequently also $f(\Sigma')$) has no point of period less than K, and still $h(\Sigma') > h(\Sigma) - \delta$. The point of the passage from X to Σ' is that by [Boyle-Krieger1993, Theorem 1.5], the periodic point condition on Σ' guarantees that the embedding

$$(f|\Sigma')^{-1} : f(\Sigma') \to \Sigma$$

can be extended to an automorphism ϕ of S. Clearly the restriction of ϕf to Σ' is the identity map. □

Theorem 3.2. *Suppose f is a surjective c.a. on N symbols and $\epsilon > 0$. Then there is an invertible c.a. ϕ such that $\nu(\phi f, S_N) > N - \epsilon$.*

Proof. If T is a mixing shift of finite type with $h(T) = \log \lambda$, then $\lim_k |\mathrm{Fix}(T^k)|^{1/k} = \lambda$. If this T is a set of fixed points for a c.a. ψ on N symbols, it follows that $\nu(\psi, S_N) \geq \lambda$. Now the theorem follows from Lemma 3.1. □

Remark 3.3. The statements of Lemma 3.1 and Theorem 3.2 remain true if ϕf is replaced by $f\phi$. One way to see this is to notice that the systems $(f\phi, S)$ and $(\phi(f\phi)\phi^{-1}, \phi S \phi^{-1}) = (\phi f, S)$ are topologically conjugate.

(2) Now we turn to algebra. Σ_N is a group under coordinatewise addition (mod N), and some c.a. are group endomorphisms of this group; these are the *linear* cellular automata whose jointly periodic points were studied in [Martin-Odlyzko-Wolfram1984] and later in a number of papers (see [Chin-Cortzen-Goldman2001, Cordovil-Dilao-daCosta1986, Jen1988, Lidman-Thomas2006, Misiurewicz-Stevens-Thomas2006, Sutner] and their references). The algebraic structure allowed a number theoretic description of the way that f-periods of jointly periodic points of S_N period n vary (irregularly) with n. We show now that when f is a linear c.a., it is easy to see that $\nu(f, S_N) = N$.

Proposition 3.4. *Suppose a c.a. map f is a surjective linear map on S_N. Then for all large primes p, $\nu_p(f, S_N) \geq N^{p-1}$. Therefore $\nu(f, S_N) = N$.*

Proof. We use an argument from the proof of a related result, Proposition 3.2 of [Boyle-Kitchens1999]. Let M be the cardinality of the kernel of f. Suppose $p > M$ and p is prime; then f must map orbits of length p to orbits of length p (otherwise, some orbit of length p would be collapsed to an orbit of length dividing p, i.e. to a fixed point, which would contradict the fact that every point has M preimages). Let H denote $P_p(S_N)$, the set of fixed points of $(S_N)^p$; H is a subgroup which is mapped into itself by f. Pick $k > 0$ such that the restriction of f to $f^k H$ is injective; then $f^k H$ is the set of points in H which are f-periodic. The kernel of f^k contains no point in an orbit of length p, so $\ker(f^k|H) \subset \mathrm{Fix}(S_N)$. Thus

$$\nu_p(f, S_N) = |f^k H| = |H/\ker(f^k|H)| \geq |H/\mathrm{Fix}(S_N)| = N^p/N = N^{p-1} \ .$$

□

(3) Algebra can be used in another way. Frank Rhodes [Rhodes1988], using properties of certain families of polynomials presenting c.a. maps, exhibited a family of noninvertible c.a. f for which there exists $k \in \mathbb{N}$ such that f is injective on $P_{kn}(S_N)$ for all $n \in \mathbb{N}$. Clearly in this case $\nu(f, S_N) = N$. We will not review that argument.

(4) We now turn to equicontinuity. A point x is equicontinuous for f if for every positive integer M there exists a positive integer K such that for all points y, if $x[-K, K] = y[-K, K]$ then $(f^n x)[-M, M] = (f^n y)[-M, M]$ for all $n > 0$. If the surjective c.a. f has x as a point of equicontinuity, and M is chosen larger than the span of the block code f, and W is the corresponding word $x[-K, K]$, then the

following holds: if z is a point in which W occurs with bounded gaps, then z is f-periodic. Thus $\lim_n (1/n) \log \nu_n(f, S_N) = \log(N)$, and moreover the convergence is exponentially fast [Blanchard-Tisseur2000]. Points of equicontinuity may occur in natural examples [Blanchard-Maass1996, Kurka1977].

For many (probably "most") surjective c.a., the criteria above are not applicable. This leads to the experimental investigations discussed in the next section, and to the possibility raised in Questions 1.2 and 1.3 of a general plenitude of jointly periodic points. Question 1.2 arises because in the experimental data, the restrictions of the c.a. f to $P_k(S_N)$ are somewhat reminiscent of a random map on a finite set. Since f is a surjective one dimensional c.a. map, there is an M such that no point has more than M preimages under f. Suppose for example k is a prime greater than M and let $\mathcal{O}_k(S_N)$ denote the set of S_N orbits of size k. Then f defines an at most M-to-1 map f_k from $\mathcal{O}_k(S_N)$ into itself, and we see a possible heuristic: (1) in the absence of some additional structure, the sequence (f_k) will reflect some properties of random maps, and (2) an "additional structure" such as existence of equicontinuity points for f will tend to produce more rather than fewer periodic points. The beautiful and extensive theory of random maps on finite sets contains precise asymptotic distributions answering various natural questions [Sachkov1997]. Here we simply note that for a random map on a set of K elements, asymptotically on the order of $\sqrt{K}$ of the elements will lie in cycles (whether the map is bounded-to-one [Grusho1972, Theorem 2] or not [Sachkov1997]), and there will be few big cycles.

The maps f_k derived from the surjective c.a. f are nonrandom not only in being bounded-to-one, but also in that most points have the minimal possible number of preimages [Hedlund1969, Kitchens1998, Lind-Marcus1995]. To the extent it matters, this seems to work in favor of the random maps heuristic behind Question 1.2. In particular, it seems that the qualification to the random maps analogy offered in [Martin-Odlyzko-Wolfram1984, p.252], regarding large in-degrees for cellular automata, does not hold for the class of surjective c.a.

4. The maps

We examine with our programs several cellular automata on N symbols, having or not having various properties as indicated below. Except for Tables 15 and 16, all c.a. examined are on $N = 2$ symbols.

The c.a. A is the addition map $x_0 + x_1 \pmod N$. This c.a. is linear, bipermutative, and everywhere N to one.

The c.a. B is $x_0 + x_1x_2$. This c.a. is left permutative, degree 1, not right closing.

The c.a. C is $B \circ B_{rev}$, where $B_{rev} = x_0x_1 + x_2$. This c.a. is degree 1, and it is nonclosing, as it is the composition of a not-left-closing c.a. and a not-right-closing c.a.

The c.a. D is the map C composed with $(S_2)^{-2}$, i.e., D is the composition of $x_0 + x_1x_2$ with $x_{-2}x_{-1} + x_0$. All periodic points for the golden mean shift (the sequences x in which the word 11 does not occur) become fixed points for D (vs. being periodic of varying periods for C).

The c.a. E is the composition A followed by B. This c.a. on $N = 2$ symbols has degree 2, and is left permutative but not right closing.

The c.a. J on 2 symbols is A precomposed with the automorphism U of S_2 which applies the flip to the symbol in the $*$ space of the frame $10*11$. This U is

$x_0 + x_{-2}(1 + x_{-1})x_1x_2$, which equals $x_0 + x_{-2}x_1x_2 + x_{-2}x_{-1}x_1x_2$. The c.a. J has degree N and is biclosing, but is neither left permutative nor right permutative.

The c.a. G is $x_{-1} + x_0x_1 + x_2$. This c.a. on 2 symbols is bipermutative, degree 2, and is not linear.

The c.a. H is the composition $A \circ A \circ U$. It has the properties of J, except that the degree is now $2^2 = 4$.

The c.a. K is the composition $B \circ U$. This c.a. is left closing degree 1; it is not left permutative and it is not right closing.

In addition we use a library of surjective span 4 and span 5 c.a. due to Hedlund, Appel and Welch, who conducted the early investigation [Hedlund-Appel-Welch1963] in which they found all surjective c.a. on two symbols of span at most five. (This was not trivial, especially in 1963, because there are 2^{32} c.a. on two symbols of span at most five.) Among these onto maps of span four, there are exactly 32 which are not linear in an end variable (i.e., neither left nor right permutative) and which send the point $\dots 0000 \dots$ to itself. These 32 are listed in Table 1. Any other span four onto map which is not linear in an end variable is one of these 32 maps g precomposed or postcomposed with the flip map $F = x_0 + 1$. Because $gF = F(Fg)F = F^{-1}(Fg)F$, the jointly periodic data for Fg and gF will be the same. Altogether, then, we can handle all surjective span 4 maps not linear in an end variable by examining 64 maps.

According to [Hedlund-Appel-Welch1963], there are 141,792 surjective c.a. of span 5. These are arranged in [Hedlund-Appel-Welch1963] into classes – linear in end variables, compositions of lower-span maps, and the remainder. The remainder class (11,388 maps) is broken down into subclasses by patterns of generation, and a less regular residual class of 200 maps. These 200 are generated by 26 maps [Hedlund-Appel-Welch1963, Table XII] and various operations. We list the codes for this irregular class of 26 maps in Table 2, and use it as a modest sample of span 5 maps.

5. FDense

The program FDense takes as its input a c.a. f, an integer $N \geq 2$, a positive integer m and a finite set $\mathcal{K}$ of positive integers k. (FDense can also handle sets of maps as inputs, producing output for all the maps, and suppressing various data.) The input f can be given by a polynomial or a tabular rule. For a given f and each k in $\mathcal{K}$, FDense determines whether the set $\text{Per}(f) \cap P_k(S_N)$ is m-dense (in which case we say that f is m-dense at k). If not, then FDense will separately list all the S_N words of length m which do not appear in any periodic point of f in $P_k(S_N)$, in a lexicographically truncated form potentially useful for seeing patterns. (For example, if m is ten and the word 011 does not occur in the examined points, then FDense would list 011* as excluded rather than listing all words of length ten beginning with 011.)

The underlying algorithm for FDense lists all words of length m and k in tagged form and operates on tags as it moves through the words of length m with f. Memory is the fundamental constraint on FDense. With m considerably smaller than k, the essential demand on memory is the tagged list of N^k words of length k. With $N = 2$, roughly $m = 13$ and $k = 27$ was a practical limit for our machine, and this was also quite slow. We restricted our investigations almost entirely to the case of $N = 2$ symbols for two reasons: with $N = 2$ we can examine longer periods; and we would be astonished to find any relation between the questions at hand and N.

The following proposition follows from the data of Tables 3 and 4.

Proposition 5.1. *For every span 4 surjective cellular automaton on two symbols, the set of jointly periodic points is (at least) 13-dense.*

In Tables 5-7, we applied FDense, for $N = 2$ symbols, to check for which $k \leq 24$ various other surjective c.a. f are 10-dense at k.

Table 5. After postcomposition with the map $A = x_0 + x_1$, the 32 onto span 4 c.a. of Table 1 remain 10-dense at some $k \leq 24$.

Table 6. The 26 irregular span 5 maps of Table 2 are 10-dense at some $k \leq 24$.

Table 7. For each of the 32 span 4 maps j of Table 1, let $p_j(x_0, x_1, x_2, x_3)$ denote its defining polynomial. Construct a c.a. f_j with defining polynomial $x_0 + p_j(x_1, x_2, x_3, x_4)$. These f_j are demonstrated to be 10-dense at some $k \leq 24$.

For the c.a. in Tables 5-7, often the least k at which 10-density is achieved lies in the range $19 - 24$. (This is the point of Table 7, as we know already from [Boyle-Kitchens1999] that the jointly periodic points of permutative c.a. are dense.) This is consistent with the heuristic that apart from possible extra structure the c.a. map on points of least period k looks something like a random map. For a random map f from a set of 2^k points into itself, on the order of $\sqrt{2^k}$ points are expected to lie in f-cycles. For $k = 20$, we have $\sqrt{2^k} = 2^{10}$. (Of course, $10 < 24/2$. A point of S_N-period 20 will contain up to 20 distinct words of length 10; the words aren't expected to occur with complete uniformity; specific codes are not random. For the heuristic of randomness, it is perhaps striking to find the rough agreement we do see.)

We also checked 10-denseness for several c.a. on 2 symbols with specified properties, described in Section 4.

Example 5.2. [Linear] The c.a. $A = x_0 + x_1$ is 10-dense at $k = 11, 13 - 24$ out of [10,24].

Example 5.3. [Permutative, not biclosing] The c.a. B is 10-dense at $k = 22 - 24$ out of [10,24]. It is 13-dense at only $k = 25$ out of [13,25].

Example 5.4. [Not closing] The c.a. C (and likewise D) is 10-dense at $k = 17 - 24$ out of [1,24], and 13-dense for $k = 23, 24$ out of [13,24].

Example 5.5. [Degree 2, biclosing, not permutative] The c.a. J is 10-dense at $k = 23 - 25$ out of [10,25]. It is 13-dense at only $k = 25$ out of [13,25].

In summary, there is reasonable supporting evidence for the Conjecture 1.1, and the counts seen seem consistent with the random maps heuristic.

6. FPeriod

Recall $P_k(S_N)$ denotes the set of points fixed by the kth power of the full shift on N symbols. Each such point x is determined by the word $x_0 x_1 \dots x_{k-1}$.

The FPeriod program takes as input a c.a. f, an integer $N \geq 2$ and a finite set of positive integers k. For each k, the program then determines data including the following (included in tables cited below):

- P:= the number of points in $P_k(S_N)$ which are periodic for f.
- L:= the length of the longest f-cycle in $P_k(S_N)$.

The program does much more; for the points in $P_k(S_N)$, it can produce a complete list of f cycle lengths and preperiods with multiplicities, and related data such as ν_k and averages. It can also do this for points in $P_k^o(S_N)$ rather than $P_k(S_N)$ (i.e. for points of least shift period k). The program also has an option for producing truncated and assembled data for a collection of maps.

The basic algorithm idea of FPeriod is the following. FPeriod takes the given c.a. f and a given shift-period length k; stores all 2^k words of length 2^k; and then changes various tags on these words as f moves through the corresponding periodic points. The tags in particular are changed to keep track of how long f iterates before returning. When the program returns to a previously visited point, it can deduce the corresponding f period and preperiod. The essential limit of FPeriod is that for large k it becomes a horrendous memory hog. We could conveniently reach period $k = 23$, and with patience we could reach $k = 25$ or 26, before our memory resources were exhausted. In practice, running the program using $N = 2$ and $k = 26$ required 1.8 gigabytes of memory.

In this section we apply FPeriod to various maps from Section 4 with specific properties, and also to many maps of span 4 and 5. The main message is that for nonlinear maps, we generally see $\nu_k(f, S_N)$ compatible with affirmative answers to Questions 1.2 and 1.3, and frequently the data suggestion strongly that the limit $\nu(f, S_N)$ is smaller than N. Below, unless otherwise indicated, f is defined on the full shift S_N with $N = 2$, and the symbol set is $\{0, 1\}$.

Table 8 [Linear]. We exhibit results for the c.a. $A = x_0 + x_1$; here $\nu_k(A, S_2)$ is large, consistent with the fact $\nu(A, S_2) = 2$.

Table 9 [Biclosing]. We exhibit results for the c.a. J, which is A composed with an invertible c.a. The composition significantly reduces the numbers ν_k.

Table 10 [Linear composed with degree 1 permutative]. We exhibit results for the c.a. E.

Table 11 [Bipermutative]. We exhibit results for the c.a. G.

Table 12 [Permutative, not biclosing]. We exhibit results for the c.a. B.

Table 13 [Closing, not permutative, not biclosing]. We exhibit results for the c.a. K.

Table 14 [Not closing]. We exhibit results for the c.a. C.

Tables 15 and 16. We give our only examples for a c.a. on more than 2 symbols (they are c.a. on 3 symbols). The pattern is the same but we are able to investigate only up to shift period 13.

Tables 17 and 18 [Span 5 irregular]. We display data for the 26 irregular maps of span 5 given in Table 2 and discussed in Section 4.

Tables 19 and 21 [Span 4]. We exhibit data for the 32 maps g of Table 1. (This addresses all span 4 surjective c.a. on 2 symbols not linear in an end variable, as discussed in Section 4.)

Tables 20 and 22. [Span 4 composed with flip]. We exhibit data for the 32 maps of Table 1 postcomposed with the flip involution $F = x_0 + 1$.

Table 23 [Permutative comparison]. ν_k^o is computed for 16 left permutative span 5 maps, to make a rough comparison of a sample of maps which are and are not linear in an end variable. We see no particular difference.

Table 24. For $B = x_0 + x_1 x_2$, complete data for B-periods with multiplicity are found by FPeriod (not FProbPeriod) for points in $P_k(S_2)$ for $k \leq 22$.

7. FProbPeriod

The k for which the program FPeriod can explore f-periodicity of points in $P_k(S_N)$ is limited on account of the memory demands of FPeriod. This begs for a probabilistic approach. For large k it is generally useless to sample points of shift period k for f-periodicity (commonly, this will be a fraction of the shift periodic points exponentially small in k). Instead, FProbPeriod randomly samples points of period k and computes for them the length of the f-cycle into which they eventually fall. This extends the range of k which can be investigated, depending on the map; for different maps we've seen practical limits at $k = 33$ to $k = 37$ (typical), to past 50 (for the linear $x_0 + x_1$ on two symbols). In any case, we can search larger k than are accessible to us with FPeriod. The program FProbPeriod again works by listing and tagging, but now only needs to keep in memory words for the points visited along an iteration. As long as the preperiod and period of the forward orbit aren't too large, the program won't crash.

The input data for FProbPeriod then are the c.a. f ; a finite set of periods k; the number N of symbols; and the number m of points to be randomly sampled for each k. The program will for each k take m random samples of points from $P_k(S_N)$, and find the corresponding periods and preperiods with multiplicity. Given k, L denotes the largest f-period found in the sample. For any sequence of samples, clearly $\limsup_k L^{1/k} \leq \limsup_k \nu_k(f, S_N) \leq N$, and inequalities must become sharp in some cases (f linear or f of finite order). Still, the data we see seems consistent with positive answers to Questions 1.2 and 1.3.

The specific maps cited below are described in Section 4.

Table 25. For sample size $m = 10$, for the (degree one, left permutative, not right closing) map $B = x_0 + x_1x_2$, the (eventual) periods are listed with their multiplicities in the sample, for $1 \leq k \leq 37$.

Table 26. For the map B, periods with multiplicity are probed for $k \leq 30$ for two samples, of size 10 and size 30. The maximum period is the same except for two values of k. By comparison with Table 24, one sees that the size 30 sample in Table 26 found the largest period except at $k = 12$ (where it found period 56 but not the maximum period 60).

Table 31. For the linear c.a. A, periods with multiplicity are probed for $k \leq 49$ for two sample sizes, 10 and 30. The results are almost identical.

Table 27. For sample size $m = 10$, for $1 \leq k \leq 37$, the numbers $L^{1/k}$ are computed for several c.a. described in Section 4: A, B, C, E, G, H, J, K. The corresponding preperiod data is displayed in *Table 28.*

Table 29. For sample size $m = 10$, for $1 \leq k \leq 32$, the sampled periods for the nonclosing c.a. C are listed with their multiplicities in the sample.

Table 30. For sample size $m = 10$, for $1 \leq k \leq 32$, the sampled periods for the nonclosing c.a. D are listed with their multiplicities in the sample.

Table 32. This table lists the preperiods found for B by FProbPeriod for the sample size 10 in the range $18 \leq k \leq 35$.

Table 33. This table lists the preperiods found for C by FProbPeriod for the sample size 10 in the range $18 \leq k \leq 35$.

For the c.a. maps f on $N = 2$ symbols explored by FProbperiod, perhaps the most striking feature observed is the exponential size of $L_k(f)$ (the length of the largest f-cycle found). For example, for the eight maps sampled in Table 27, at $k =$ 37 $L_k(f)$ is approximately α^k with $\alpha = 1.49, 1.60, 1.55, 1.46, 1.30, 1.44, 1.45, 1.57.$

These long cycles are compatible with the heuristic that we should for arbitrarily large k see at least as much periodicity as we would expect from a random map (but not necessarily more). A random map on 2^k points would for large k produce a longest cycle of size on the order of $\sqrt{2^k} \approx 1.4^k$.

8. Online tables

The tables referred to in the paper are available in the online supplement to this paper at the Experimental Mathematics website (http://www.expmath.org/expmath/volumes/VOLX/VOLX.ISS
and are also at the website of the first named author. The tables are organized in four groups:

- Tables of some span 4 and 5 c.a. (Tables 1-2)
- FDense Tables (Tables 3-7)
- FPeriod Tables (Tables 8-24)
- FProbPeriod Tables (Tables 25-33)

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Mike Boyle, Department of Mathematics, University of Maryland, College Park, MD 20742-4015, U.S.A.

E-mail address: `mmb@math.umd.edu`

URL: `www.math.umd.edu/∼mmb`

Bryant Lee, Department of Mathematics, University of Maryland, College Park, MD 20742-4015, U.S.A.

E-mail address: `blee3@umd.edu`