

PATH METHODS FOR STRONG SHIFT EQUIVALENCE OF POSITIVE MATRICES

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ABSTRACT. In the early 1990's, Kim and Roush developed path methods for establishing strong shift equivalence (SSE) of positive matrices over a dense subring \mathcal{U} of \mathbb{R} . This paper gives a detailed, unified and generalized presentation of these path methods. New arguments which address arbitrary dense subrings \mathcal{U} of \mathbb{R} are used to show that for any dense subring \mathcal{U} of \mathbb{R} , positive matrices over \mathcal{U} which have just one nonzero eigenvalue and which are strong shift equivalent over \mathcal{U} must be strong shift equivalent over \mathcal{U}_+ . In addition, we show matrices on a path of positive shift equivalent real matrices are SSE over \mathbb{R}_+ ; positive rational matrices which are SSE over \mathbb{R}_+ must be SSE over \mathbb{Q}_+ ; and for any dense subring \mathcal{U} of \mathbb{R} , within the set of positive matrices over \mathcal{U} which are conjugate over \mathcal{U} to a given matrix, there are only finitely many SSE- \mathcal{U}_+ classes.

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1. INTRODUCTION

The classification problem for shifts of finite type (SFTs) remains a central open problem for symbolic dynamics. In the foundational work of Williams [30, 31] over forty years ago, the problem was recast as the following question: when are two matrices strong shift equivalent (SSE) over \mathbb{Z}_+ ? (The definition of SSE for matrices

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over a semiring is recalled below in Definition 2.1. In this paper, rings and semirings are always assumed to contain 1.) Since then, SSE (involving other semirings) has been used for classification of other symbolic dynamical systems: for example, SFTs with Markov measure [22], using matrices over Laurent polynomials; SFTs with a finite group action [8], using matrices over the integral group ring of a finite group [5]; and sofic shifts [4, 10, 13], using a more complicated ring. Matsumoto has extended the ideas of SSE to a classification setting for arbitrary subshifts [23, 24].

The original, notoriously difficult question of Williams for \mathbb{Z}_+ remains unanswered, and this is also a barrier to understanding the other classifications. One probe into this problem is to consider SSE over \mathcal{U}_+ for primitive matrices over a dense subring \mathcal{U} of \mathbb{R} .

Over a series of papers [14, 15, 16, 17] ending in 1992, Kim and Roush introduced path methods for the study of strong shift equivalence of positive matrices over the reals and certain subrings of it. One highlight of this work was the following theorem. For \mathcal{U} any subfield of \mathbb{R} , if A and B are square matrices over \mathcal{U}_+ which have eventual rank 1 (all large powers have rank 1) and have the same nonzero eigenvalue, then A and B are SSE over \mathcal{U}_+ . (As in Remark 6.5, there are in a sense no general results for greater eventual rank.) We extend this theorem to arbitrary dense subrings of \mathbb{R} , under the additional necessary condition that A and B are SSE over the ring \mathcal{U} . An additional condition cannot be avoided: for general \mathcal{U} , matrices with eventual rank 1 and the same nonzero eigenvalue need not be even shift equivalent over \mathcal{U} to a 1×1 matrix (see Remark 6.8). Whether the assumption of SSE over \mathcal{U} is equivalent to the more tractable condition of shift equivalence over \mathcal{U} remains an open question. However, our proof requires the assumption of SSE, not just SE, over \mathcal{U} .

The central result of the path methods development was a Path Theorem for \mathbb{R} : matrices on a path of positive conjugate (similar) matrices must be SSE over \mathbb{R}_+ . In this paper, we prove a generalized Path Theorem (5.10) which has application to arbitrary dense subrings of \mathbb{R} . We also show that matrices on a path of positive matrices shift equivalent over \mathbb{R} must be SSE over \mathbb{R}_+ . This is a consequence of a more technical statement, the Connection Theorem (7.3), which relies in turn on a result which is pure linear algebra (Theorem D.2).

One indication of the power of the path method comes from the corollary due to Chuysurichay (Theorem 5.12): the set of positive matrices in a given conjugacy (similarity) class over \mathbb{R} contains only finitely many SSE- \mathbb{R}_+ classes. This holds even though (as shown by Chuysurichay) it may be impossible to connect matrices in the class with SSEs with uniformly bounded lags (see Remark 5.13). Using our Path Theorem, we generalize this finiteness result to arbitrary dense subrings of \mathbb{R} (Theorem 5.16).

Using the Connection Theorem, we are able to show that positive real matrices SSE over \mathbb{R}_+ are SSE over \mathbb{R}_+ through positive matrices (Theorem 8.1). As a consequence, primitive positive trace matrices over a subfield \mathcal{U} of \mathbb{R} which are SSE over \mathbb{R}_+ must also be SSE over \mathcal{U}_+ (Theorem 8.2).

Altogether, for positive matrices SSE over a dense subring \mathcal{U} of \mathbb{R} , the current paper reduces the gap between SSE over \mathbb{R}_+ and SSE over \mathcal{U}_+ , and provides further evidence for the utility of investigating SSE of positive matrices over \mathbb{R}_+ . This is a problem to which more standard mathematics (e.g. fiber bundles, linear algebra) can be applied, as seen in [13, 14, 15, 16, 17] and the current paper. So, we suggest

splitting into three parts the problem of understanding when two positive matrices over a dense subring \mathcal{U} of \mathbb{R} are SSE over \mathcal{U}_+ :

- (1) Assuming A and B SSE over \mathbb{R} , prove they are SSE over \mathbb{R}_+ .
- (2) Assuming A and B SSE over \mathbb{R}_+ and over \mathcal{U} , determine whether they are SSE over \mathcal{U}_+ .
- (3) Understand the refinement of SE over \mathcal{U} by SSE over \mathcal{U} .

We now say a little about the organization of the paper.

In Section 2, we explain the decomposition of SSE into row splittings, column splittings and diagonal refactorizations, and provide some basic technical results essential for the sequel. The results of this section hold over quite general rings, and refine the basic Williams theory.

In Section 3, given positive matrices A, B which are SSE over \mathcal{U} , we produce positive matrices A', B' which are conjugate over \mathcal{U} such that A is SSE- \mathcal{U}_+ to A' and B is SSE- \mathcal{U}_+ to B' . (This is the step for which we need matrices SSE- \mathcal{U} , not just SE- \mathcal{U} .)

In Section 4, we study $\text{Cent}_{\mathbb{R}}(A)$, the group of invertible real matrices which commute with a given $n \times n$ real matrix A , and its group of connected components, $\pi_0(\text{Cent}_{\mathbb{R}}(A))$. This group plays a key role in the formulation of obstructions to applying the Path Theorem to produce SSE- \mathcal{U}_+ .

In Section 5, we prove the Path Theorem 5.10 and some consequences. In Section 6, we prove the eventually rank 1 results. In Section 7, we prove the Connection Theorem. A large part of the proof is an independent result in linear algebra, which we relegate to Appendix D. In Section 8, we prove in particular that positive rational matrices SSE over \mathbb{R}_+ must be SSE over \mathbb{Q}_+ . This is some supporting evidence for the conjecture [2, Conj. 5.1] that positive rational matrices shift equivalent over \mathbb{Q}_+ are SSE over \mathbb{Q}_+ .

This paper is entirely devoted to matrices. For background on shifts of finite type and symbolic dynamics, see [20, 19].

2. ELEMENTARY SPLITTING AND STRONG SHIFT EQUIVALENCE

Bob Williams introduced shift equivalence and strong shift equivalence in his paper [30], which is the foundation of all future work on the topic. One of the fundamental contributions was a decomposition of an elementary strong shift equivalence using even more fundamental relations, splittings and amalgamations. In [30], Williams considered matrices over \mathbb{Z}_+ and $\{0, 1\}$. For our work with unital nondiscrete subrings of \mathbb{R} , we need some refinements to this work.

Definition 2.1. Let \mathcal{U} be a subset of a semiring containing 0 and 1 (additive and multiplicative identities). Matrices A, B are *elementary strong shift equivalent over \mathcal{U}* (ESSE- \mathcal{U}) if there exist matrices R, S over \mathcal{U} such that $A = RS$ and $B = SR$. Matrices A, B are *strong shift equivalent over \mathcal{U}* (SSE- \mathcal{U}) if there exist matrices A_0, A_1, \dots, A_ℓ , and for $1 \leq i \leq \ell$ matrices R_i, S_i over \mathcal{U} such that $A_{i-1} = R_i S_i$ and $A_i = S_i R_i$, with $A_0 = A$ and $A_\ell = B$. In this case the string (R_i, S_i) , $1 \leq i \leq \ell$, is a *strong shift equivalence of lag ℓ* from A to B .

Although we do not use shift equivalence before Section 7, to clarify ideas we recall its basic features now.

Definition 2.2. Let \mathcal{U} be a subset of a semiring containing 0 and 1 (additive and multiplicative identities). Matrices A, B are *shift equivalent over \mathcal{U}* (SE- \mathcal{U}) if there

exist matrices R, S over \mathcal{U} and $\ell \in \mathbb{N}$ such that the following hold:

$$\begin{aligned} A^\ell &= RS & B^\ell &= SR \\ AR &= RB & BS &= SA . \end{aligned}$$

Always, $\text{SE-}\mathcal{U}$ implies $\text{SSE-}\mathcal{U}$. The converse is true if \mathcal{U} is a Dedekind domain [3] (e.g., a field or \mathbb{Z} , [7, 31]). For primitive matrices A, B over a subring of \mathbb{R} : A, B are $\text{SE-}\mathcal{U}$ if and only if A, B are $\text{SE-}\mathcal{U}_+$. Over \mathcal{U} a subfield of \mathbb{R} , matrices are shift equivalent if and only if the nonsingular parts of their Jordan forms are the same. There is a ‘‘conceptual’’ version of shift equivalence, in terms of isomorphism of associated dimension modules.

Williams asked whether the relatively tractable relation $\text{SE-}\mathbb{Z}_+$ implies $\text{SSE-}\mathbb{Z}_+$. Working within Wagoner’s algebraic topological framework for the classification problem [28], Kim and Roush gave examples of primitive matrices over \mathbb{Z} which are SSE over \mathbb{Z} (equivalently, shift equivalent over \mathbb{Z}) but not SSE over \mathbb{Z}_+ [18]. There are also examples of positive matrices over a dense subring \mathcal{U} of \mathbb{R} which are SSE over \mathcal{U} but are not SSE over \mathcal{U}_+ (see Remark 5.14). A feature of Wagoner’s framework is that it is built up out of elementary SSE s, not out of SE . We will see the same feature in the proof of Theorem 3.1. For a ring \mathcal{U} , can be useful to take $\text{SSE-}\mathcal{U}$ as a hypothesis, and leave the question of whether $\text{SE-}\mathcal{U}$ implies $\text{SSE-}\mathcal{U}$ as a separate issue.

We turn away now from shift equivalence, until Section 7. For a ring \mathcal{U} ,

Definition 2.3. An *amalgamation matrix* is a matrix with entries from $\{0, 1\}$ such that every row has exactly one 1 and every column has at least one 1. A *subdivision matrix* is the transpose of an amalgamation matrix.

Definition 2.4. An *elementary row splitting* is an elementary strong shift equivalence $UX = A, XU = C$ in which U is a subdivision matrix. In this case, C is an *elementary row splitting* of A , and A is an *elementary row amalgamation* of C .

Definition 2.5. An *elementary column splitting* is an elementary strong shift equivalence $XV = A, VX = C$ in which V is an amalgamation matrix. In this case, C is an *elementary column splitting* of A , and A is an *elementary column amalgamation* of C .

Given $a_1 + a_2 + a_3 = a$, $b_1 + b_2 + b_3 = b$, $c_1 + c_2 = c$, $d_1 + d_2 = d$, here is an elementary row splitting C of A :

$$\begin{aligned} UX &= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \\ XU &= \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 & a_1 & b_1 & b_1 \\ a_2 & a_2 & a_2 & b_2 & b_2 \\ a_3 & a_3 & a_3 & b_3 & b_3 \\ c_1 & c_1 & c_1 & d_1 & d_1 \\ c_2 & c_2 & c_2 & d_2 & d_2 \end{pmatrix} = C \end{aligned}$$

For an elementary row splitting, rows of A are split as sums of rows (as described by X), and then columns of X are ‘‘copied’’ in such a way that indices of rows in

C with the same “parent” row in A have equal columns in C . We say a row in C in A is *sitting above* its parent row.

Similarly, here is an example of an elementary column splitting.

$$YV = \begin{pmatrix} -3.4 & 1.4 & 5 \\ \pi & -\pi & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 0 & 6 \end{pmatrix} = A$$

$$VY = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3.4 & 1.4 & 5 \\ \pi & -\pi & 6 \end{pmatrix} = \begin{pmatrix} -3.4 & 1.4 & 5 \\ -3.4 & 1.4 & 5 \\ \pi & -\pi & 6 \end{pmatrix}$$

Here, columns 1 and 2 of VY are sitting above column 1 of A in the column splitting.

Definition 2.6. A matrix is *nondegenerate* if it has no zero row and it has no zero column.

Definition 2.7. A *diagonal refactorization over a semiring \mathcal{U}* is an elementary strong shift equivalence over \mathcal{S} of the form $A = DX$, $B = XD$, where D is nondegenerate diagonal over \mathcal{S} . In this case, A is a *diagonal refactorization of B* (and vice versa).

We now recall the canonical factorization of a nondegenerate matrix introduced by Williams [30]).

Suppose M is a nondegenerate matrix over a semiring \mathcal{U} containing $\{0, 1\}$, with rows indexed by the set \mathcal{I} and columns indexed by the set \mathcal{J} . Let \mathcal{E} be the set of pairs (i, j) such that $M(i, j) \neq 0$. Let U_M be the $\mathcal{I} \times \mathcal{E}$ subdivision matrix such that $U_M(i', (i, j)) = 1$ iff $i' = i$. Let V_M be the $\mathcal{E} \times \mathcal{J}$ amalgamation matrix such that $V_M((i, j), j') = 1$ iff $j = j'$. Let D_M be the $\mathcal{E} \times \mathcal{E}$ diagonal matrix such that $D_M((i, j), (i, j)) = M(i, j)$. Then

$$M = U_M D_M V_M .$$

Because M is nondegenerate, U_M and V_M are defined (e.g., given i there is at least one j such that $D(i, j) \neq 0$, so row i of U_M has at least one 1), and D has nonzero diagonal entries.

There is a graphical interpretation of the factorization $M = U_M D_M V_M$. The set \mathcal{E} can be viewed as the set of edges of a directed graph, in which there is an edge from i to j if $M(i, j)$ is nonzero. The matrices U_M and V_M attach (respectively) initial and terminal vertices to edges, and D_M records the entry of M labeling the edge.

Definition 2.8. We call the factorization $M = U_M D_M V_M$ above of a nondegenerate matrix M the *Williams factorization of M* . It is well defined up to the choice of ordering of indices used for D_M (and thus U_M and V_M).

If M is a nondegenerate matrix and $M = UDV$ with U subdivision, D nondegenerate diagonal and V amalgamation, then $M = UDV$ must be the Williams factorization described above.

We may avoid the complications of defining a factorization for degenerate matrices, on account of the following proposition.

Proposition 2.9. *Suppose \mathcal{U} is a ring which is torsion free as an additive group. Suppose nondegenerate matrices A and B are SSE over \mathcal{U} . Then they are SSE through a chain of ESSEs $A_{i-1} = R_i S_i$, $A_{i+1} = S_i R_i$ such that all the matrices A_i, R_i, S_i are nondegenerate.*

The proof of Proposition 2.9 is a digression, and we give it in Appendix A.

Proposition 2.10. *Suppose $A = RS, B = SR$ is an elementary strong shift equivalence over a semiring \mathcal{U} containing $\{0, 1\}$; \mathcal{U} has no zero divisors; and the matrices A, B are nondegenerate. Then there are nondegenerate matrices C_1, C_2, D over \mathcal{U} such that D is diagonal and*

- (1) C_1 is an elementary row splitting of A
- (2) There is a matrix X over \mathcal{U} such that $and $(so, C_1 is a diagonal refactorization of C_2)$$
- (3) C_2 is an elementary column splitting of B .

Proof. Using the Williams factorization above, we have

$$\begin{aligned} A &= (U_R D_R V_R) (U_S D_S V_S) \\ B &= (U_S D_S V_S) (U_R D_R V_R) . \end{aligned}$$

Define

$$\begin{aligned} C_1 &= (D_R V_R U_S D_S V_S) U_R , & X_1 &= D_R V_R U_S D_S V_S \\ C_2 &= V_R (U_S D_S V_S U_R D_R) , & X_2 &= U_S D_S V_S U_R D_R . \end{aligned}$$

Set $D = D_R$ and $X = V_R U_S D_S V_S U_R$. Then $DX = C_1$ and $XD = C_2$, proving (2). Also,

$$\begin{aligned} A &= U_R X_1 & B &= X_2 V_R \\ C_1 &= X_1 U_R & C_2 &= V_R X_2 . \end{aligned}$$

This proves (1) and (3). It remains to prove the nondegeneracy claims.

The matrix $D = D_R$ is nondegenerate by construction. The matrix X_2 has no zero row, because $B = X_2 V_R$ has no zero row. The matrix $(U_S D_S V_S)$ has no zero column because A has no zero column. Because U_R is a subdivision matrix, the matrix $(U_S D_S V_S) U_R$ then has no zero column. Because there are no zero divisors, the matrix $(U_S D_S V_S) U_R D_R = X_2$ has no zero column. Thus X_2 is nondegenerate. Because V_R is an amalgamation matrix, the matrix $C_2 = V_R X_2$ is also nondegenerate. Similarly, C_1 is nondegenerate. This proves the proposition. \square

Remark 2.11. If (for example) \mathcal{U} is a subring of the reals, then in Proposition 2.10 the matrix D^{-1} is defined over \mathbb{R} and the matrices $D^{-1}C_1$ and $C_2 D^{-1}$ have entries in \mathcal{U} . Then we can summarize the proposition with a diagram

$$(2.1) \quad \begin{array}{ccc} & C_1 & \xrightarrow{(D, D^{-1}C_1)} D^{-1}C_1 D \\ (X, U) \swarrow & & \searrow (V, Y) \\ A & & B \end{array}$$

in which an arrow labelled (J, K) from M to N represents an elementary strong shift equivalence $M = JK, N = KJ$; U is a subdivision matrix; and V is an amalgamation matrix. C_1 is an elementary row splitting of A and $D^{-1}C_1 D$ is an elementary column splitting of B .

For matrices over $\{0, 1\}$, the next lemma is well known ([25], [20, Theorem 2.1.14]), and can be interpreted as a fiber product statement.

Lemma 2.12 (Fiber Lemma). *Suppose over a ring \mathcal{U} there is an elementary row splitting of A to a nondegenerate C_1 and an elementary column splitting of A to a nondegenerate C_2 . Then there is a nondegenerate matrix F such that over \mathcal{U} there is an elementary column splitting of C_1 to F and an elementary row splitting of C_2 to F .*

If all entries of C_1 and C_2 are nonnegative, or positive, entries in a nondiscrete unital subring \mathcal{U} of \mathbb{R} , then all entries of F can be chosen to have nonnegative, or positive, entries in \mathcal{U} .

Proof. Let \mathcal{I} denote the set indexing the rows and columns of A . For $s = 1, 2$ let \mathcal{I}_s be the index set for the rows and columns of C_s . The index set for the rows and columns of F will be the set

$$\mathcal{V} := \{(i_1, i_2) \in \mathcal{I}_1 \times \mathcal{I}_2 : \overline{i_1} = \overline{i_2} = i\}$$

where $\overline{i_s}$ denotes the element of \mathcal{I} associated to i_s under the given elementary splitting of A to C_s . For $i \in \mathcal{I}$, let $\mathcal{V}(i) = \{(i_1, i_2) \in \mathcal{V} : \overline{i_1} = \overline{i_2} = i\}$. Let $F_{\langle i,j \rangle}$ denote the submatrix of F with index set $\mathcal{V}(i) \times \mathcal{V}(j)$. Given $i \in \mathcal{I}$, we let $\mathcal{I}_s(i)$ denote the set of indices i_s in \mathcal{I}_s such that $\overline{i_s} = i$.

We will define F by defining $F_{\langle i,j \rangle}$ for each i, j . So, consider now i, j from \mathcal{I} . For notational simplicity, suppose for the definition of $F_{\langle i,j \rangle}$ that $\mathcal{I}_1(i) = \{1, \dots, m\}$ and $\mathcal{I}_2(j) = \{1, \dots, n\}$. We will define an $m \times n$ matrix $M = M_{\langle i,j \rangle}$ and then set $F_{\langle i,j \rangle}((i_1, i_2), (j_1, j_2)) = M(i_1, j_2)$.

Let a denote $A(i, j)$. By the nature of row splitting, there is a vector $\alpha = (\alpha_1, \dots, \alpha_m)$ over \mathcal{U} such that $C_1(s, k) = \alpha_s$ for every $k \in \mathcal{I}_1(j)$ and $1 \leq s \leq m$. Likewise, there is a vector $\beta = (\beta_1, \dots, \beta_n)$ over \mathcal{U} such that $C_2(k, t) = \beta_t$ for every $k \in \mathcal{I}_2(i)$ and $1 \leq t \leq n$. Also, $\sum_s \alpha_s = a = \sum_t \beta_t$.

We now arrange that the vector of row sums of M is α and the vector of column sums of M is β . (In the special case that \mathcal{U} is a field, for $a \neq 0$ we could simply set $M(s, t) = (1/a)\alpha(s)\beta(t)$.) If $m = 1$, we necessarily set $M(1, t) = \beta(t)$, $1 \leq t \leq n$. If $n = 1$, we likewise set $M(s, 1) = \alpha(s)$, $1 \leq s \leq m$. If $m = n = 1$, the two definitions coincide. If m and n are greater than 1, pick an $(m-1) \times (n-1)$ matrix N over \mathcal{U} and for $1 \leq s < m$ and $1 \leq t < n$ define $M(s, t) = N(s, t)$. Then for $1 \leq s < m$, define $M(s, n)$ so that the s th row sum is $\alpha(s)$, and for $1 \leq t < n$ define the entries $M(m, t)$ so that the t th column sum is $\beta(j_t)$. These additional entries must lie in the ring \mathcal{U} . Finally define $M(m, n)$ so that the sum of the entries of M is a . Necessarily $M(m, n)$ is in \mathcal{U} . The m th row sum of M is $\alpha(m)$ because it equals a minus the sum of the other row sums $\alpha(1), \dots, \alpha(m-1)$. Similarly the n th column sum of M is $\beta(n)$.

In the case that C_1 and C_2 have nonnegative real entries and $m > 1$ and $n > 1$, if $a = 0$ then set $M = 0$. If $a > 0$, then for notational convenience suppose $\alpha_m \beta_n > 0$. Then choose N above such that $N(s, t) = 0$ whenever $\alpha(s)\beta(t) = 0$ and otherwise

$$0 < (1/a)\alpha(s)\beta(t) - N(s, t) < \epsilon$$

where ϵ is small enough to guarantee that $M(m, n) > 0$. Then M will be nonnegative, and M will be positive if α and β are positive. This finishes the definition of M and F .

Now define a $\mathcal{V} \times \mathcal{I}_1$ amalgamation matrix V and an $\mathcal{I}_1 \times \mathcal{V}$ matrix X by the rules

$$\begin{aligned} V\left((i_1, i_2), i_3\right) &= 1 \quad \text{iff } i_1 = i_3 \\ X\left(i_1, (j_1, j_2)\right) &= M_{\langle i, j \rangle}(i_1, j_2), \quad \text{where } (i, j) = (\overline{i_1}, \overline{j_2}). \end{aligned}$$

Then

$$\begin{aligned} (XV)(i_1, j_1) &= \sum_{\{j_2: (j_1, j_2) \in \mathcal{V}\}} X(i_1, (j_1, j_2))V((j_1, j_2), j_1) \\ &= \sum_{\{j_2: \overline{j_1} = \overline{j_2}\}} M_{\langle i, j \rangle}(i_1, j_2), \quad \text{where } (i, j) = (\overline{i_1}, \overline{j_2}), \\ &= \alpha(i_1) = C_1(i_1, j_1). \end{aligned}$$

Similarly,

$$\begin{aligned} (VX)((i_1, i_2), (j_1, j_2)) &= X(i_1, (j_1, j_2)) \\ &= M_{\langle i, j \rangle}(i_1, j_2), \quad \text{where } (i, j) = (\overline{i_1}, \overline{j_2}), \\ &= F((i_1, i_2), (j_1, j_2)). \end{aligned}$$

Thus $C_1 = VX$ and $F = XV$, and F is an elementary column splitting of C_1 .

Likewise, F is an elementary row splitting of C_2 . Define an $\mathcal{I}_2 \times \mathcal{V}$ subdivision matrix U and a $\mathcal{V} \times \mathcal{I}_2$ matrix Y by the rules

$$\begin{aligned} U(i_3, (i_1, i_2)) &= 1 \quad \text{iff } i_3 = i_2 \\ Y((i_1, i_2), j_2) &= M_{\langle i, j \rangle}(i_1, j_2), \quad \text{where } (i, j) = (\overline{i_1}, \overline{j_2}). \end{aligned}$$

Then $F = YU$ and $C = UY$, by a similar computation.

Finally, suppose C_1 and C_2 are nondegenerate. Then F has no zero column (being a row splitting of C_2) and F has no zero row (being a column splitting of C_1), so F is nondegenerate. \square

Lemma 2.13. *Suppose \mathcal{U} is a unital ring, A and B are $n \times n$ matrices over \mathcal{U} , and there is a nondegenerate diagonal matrix D and a matrix C such that $A = DC$ and $B = CD$.*

Then there are matrices A', B' over \mathcal{U} such that A' is an elementary row splitting of A , B' is an elementary column splitting of B , and A' is conjugate over \mathcal{U} to B' . If A and B are nondegenerate and the ring \mathcal{U} has no zero divisors, then the matrices A', B' can be chosen nondegenerate.

Proof. If $D = I_n$, we are done, so suppose not. For notational simplicity, suppose there is a positive integer k such that $D(i, i) = 1$ iff $i > k$. Suppose $k < n$. Let E denote the $k \times k$ upper left corner of D . Then in block form, for some matrices C_i over \mathcal{U} (with $1 \leq i \leq 4$ and C_1 $k \times k$) we have

$$A = \begin{pmatrix} EC_1 & EC_2 \\ C_3 & C_4 \end{pmatrix}, \quad B = \begin{pmatrix} C_1E & C_2 \\ C_3E & C_4 \end{pmatrix}.$$

An elementary row splitting of A to an $(n+k) \times (n+k)$ matrix A' is given by

$$\begin{aligned} A &= \begin{pmatrix} I_k & I_k & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ (-I_k + E)C_1 & (-I_k + E)C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} EC_1 & EC_2 \\ C_3 & C_4 \end{pmatrix} \\ A' &= \begin{pmatrix} C_1 & C_2 \\ (-I_k + E)C_1 & (-I_k + E)C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} I_k & I_k & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix} \\ &= \begin{pmatrix} C_1 & C_1 & C_2 \\ (-I_k + E)C_1 & (-I_k + E)C_1 & (-I_k + E)C_2 \\ C_3 & C_3 & C_4 \end{pmatrix}. \end{aligned}$$

An elementary column splitting of B to an $(n+k) \times (n+k)$ matrix B' is given by

$$\begin{aligned} B &= \begin{pmatrix} C_1 & C_1(-I_k + E) & C_2 \\ C_3 & C_3(-I_k + E) & C_4 \end{pmatrix} \begin{pmatrix} I_k & 0 \\ I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} C_1 E & C_2 \\ C_3 E & C_4 \end{pmatrix} \\ B' &= \begin{pmatrix} I_k & 0 \\ I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} C_1 & C_1(-I_k + E) & C_2 \\ C_3 & C_3(-I_k + E) & C_4 \end{pmatrix} \\ &= \begin{pmatrix} C_1 & C_1(-I_k + E) & C_2 \\ C_1 & C_1(-I_k + E) & C_2 \\ C_3 & C_3(-I_k + E) & C_4 \end{pmatrix}. \end{aligned}$$

Define

$$W = \begin{pmatrix} 0 & I_k & 0 \\ I_k & E - 2I_k & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix}, \text{ with inverse } W^{-1} = \begin{pmatrix} 2I_k - E & I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix}.$$

A computation shows $A'W = WB'$. If A and B are nondegenerate and \mathcal{U} has no zero divisors, then the constructed matrices A' and B' are nondegenerate. This finishes the proof for the case $k < n$. If $k = n$, then simply remove block rows and columns through C_4 from the proof above, and repeat the proof with D in place of E . \square

3. FROM STRONG SHIFT EQUIVALENCE TO CONJUGACY

Let \mathcal{U} be a nondiscrete unital subring of \mathbb{R} . Two $n \times n$ matrices A and B with entries in \mathcal{U} are *conjugate over \mathcal{U}* , or *similar over \mathcal{U}* , if there exists W in $\text{GL}(n, \mathcal{U})$ such that $W^{-1}AW = B$.

The purpose of this section is to prove the following theorem.

Theorem 3.1. *Let \mathcal{U} be a nondiscrete unital subring of \mathbb{R} . Suppose A, B are positive matrices over \mathcal{U} and are strong shift equivalent over \mathcal{U} . Then A and B are strong shift equivalent over \mathcal{U}_+ to positive matrices which are conjugate over \mathcal{U} .*

Moreover, the conjugating matrix can be chosen to have positive determinant and to send positive eigenvectors to positive eigenvectors.

We begin with the main lemma. We use the following notation: I_k denotes the $k \times k$ identity matrix.

Lemma 3.2 (Splitting Lemma). *Let \mathcal{U} be a nondiscrete unital subring of \mathbb{R} . Suppose the following:*

- A and C are $n \times n$ matrices over \mathcal{U}
- W is a matrix in $\text{GL}(n, \mathcal{U})$ such that $W^{-1}AW = C$
- C' is obtained from C by a finite sequence of row splittings over \mathcal{U} .

Then the following hold.

- (1) There is a matrix A' conjugate over \mathcal{U} to C' such that A' is obtained from A by a finite sequence of row splittings over \mathcal{U} ; and such that, if A and C' are nondegenerate, then A' is nondegenerate.
- (2) If A is a positive matrix, then there is a positive matrix A^+ over \mathcal{U} such that A^+ is obtained from A by a finite sequence of row splittings of positive matrices over \mathcal{U}_+ , and A^+ is conjugate over \mathcal{U} to a matrix of the form $\begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix}$.

The lemma statement is also true with “row” replaced by “column”.

Proof. Any row splitting to a larger matrix is a composition of row splittings which increase the matrix size by exactly one. So, we have some positive integer ℓ and a finite sequence of elementary row splittings of matrices C_i to C_{i+1} , $0 \leq i < \ell$, with C_{i+1} obtained by splitting one row of C_i to two rows, and with $C = C_0$ and $C' = C_\ell$.

Proof of Claim (1) We first consider the case that $\ell = 1$. For notational convenience, suppose row n of C is split into rows n and $n + 1$ of C' . For any matrix B , we let $B_{\text{row}(i)}$ denote its i th row. We have matrices

$$X' = \begin{pmatrix} C_{\text{row}(1)} \\ \vdots \\ C_{\text{row}(n-1)} \\ s' \\ t' \end{pmatrix}, \quad U' = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

such that $U'X' = C$ and $X'U' = C'$. Set $t = t'W^{-1}$ and set $s = A_{\text{row}(n)} - t$ and define the $(n + 1) \times n$ matrix

$$Y' = \begin{pmatrix} A_{\text{row}(1)} \\ \vdots \\ A_{\text{row}(n-1)} \\ s \\ t \end{pmatrix}.$$

Then $U'Y' = A$. Define $A' = Y'U'$, an elementary row splitting of A . Let E be the $(n + 1) \times (n + 1)$ matrix equal to I_{n+1} except that $E(n, n + 1) = -1$. Then we have matrix equations (in block forms)

$$\begin{aligned} E^{-1}A'E &= \begin{pmatrix} A & 0 \\ t & 0 \end{pmatrix} \\ E^{-1}C'E &= \begin{pmatrix} C & 0 \\ t' & 0 \end{pmatrix} \\ \begin{pmatrix} A & 0 \\ t & 0 \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C & 0 \\ t' & 0 \end{pmatrix}. \end{aligned}$$

Therefore A' is conjugate over \mathcal{U} to C' .

At the inductive step, going from $\ell - 1$ to ℓ , we apply the same argument to matrices $A'_{\ell-1}$ and $C'_{\ell-1}$ given by the induction hypothesis.

Now suppose A is nondegenerate. Then no sequence of row splittings of A can produce a matrix with a zero column. If C' is nondegenerate, then we can choose all those row splittings C_i to C_{i+1} , splitting some row as a sum $s_i + t_i$, such that $s_i \neq 0 \neq t_i$. Then the construction, splitting A'_i to A'_{i+1} , never introduces a zero row, and in the end A' will be nondegenerate. This completes the proof of (1).

Proof of Claim (2). As in part (1), we first consider the case $\ell = 1$. Let s', t', s, t be as in part (1). Define matrices

$$U = \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad X'' = \begin{pmatrix} C_{\text{row}(1)} \\ \vdots \\ C_{\text{row}(n-1)} \\ s' \\ t' \\ 0 \end{pmatrix}, \quad Y'' = \begin{pmatrix} A_{\text{row}(1)} \\ \vdots \\ A_{\text{row}(n-1)} \\ s \\ t \\ 0 \end{pmatrix}.$$

Set $A'' = Y''U$. Let F be the $(n+2) \times (n+2)$ matrix equal to I_{n+2} except that $F(n, n+1) = F(n, n+2) = -1$. Then

$$F^{-1}A''F = \begin{pmatrix} A & 0 & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F^{-1}C''F = \begin{pmatrix} C & 0 & 0 \\ t' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $F^{-1}C''F$ (and therefore A'') is conjugate over \mathcal{U} to the $(n+2) \times (n+2)$ matrix $\begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix}$.

It remains to conjugate A'' over \mathcal{U} to a matrix A^+ which is the required row splitting of A . For this we will pick a suitable invertible 3×3 matrix M with all column sums equal to 1, define W^+ to be $\begin{pmatrix} I_{n-1} & 0 \\ 0 & M \end{pmatrix}$ and set A^+ equal to $W^+A''(W^+)^{-1}$.

Because M has all column sums 1 (i.e. fixes the row vector with every entry 1), it follows that M^{-1} has all column sums 1, and therefore $U(W^+)^{-1} = U$. Consequently,

$$A^+ = W^+Y''U(W^+)^{-1} = W^+Y''U = \begin{pmatrix} I_{n-1} & 0 \\ 0 & M \end{pmatrix} Y''U.$$

The matrix M will have the form

$$(3.1) \quad \begin{pmatrix} x_1 & x_1 + \epsilon_1 & z_1 \\ x_2 & x_2 + \epsilon_2 & z_2 \\ 1 - (x_1 + x_2) & 1 - (x_1 + x_2) - (\epsilon_1 + \epsilon_2) & 1 - (z_1 + z_2) \end{pmatrix}$$

and therefore the bottom three rows of W^+Y'' will equal

$$(3.2) \quad M \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} x_1(s+t) + \epsilon_1 t \\ x_2(s+t) + \epsilon_2 t \\ [1 - (x_1 + x_2)](s+t) - [\epsilon_1 + \epsilon_2]t \end{pmatrix}.$$

These three rows sum to $s + t$, which is row n of A . Thus $U(W^+Y'') = A$, and $A^+ = (W^+Y'')U$ is an elementary row splitting of A .

We now complete the definition of M . Given $\gamma > 0$, pick positive numbers x_1, x_2, ϵ_1 from \mathcal{U} with $|x_1 - 1/3|, |x_2 - 1/3|$ and ϵ_1 all smaller than γ . Pick $K \in \mathbb{N}$ such that $K\epsilon_1 \leq 1 < (K+1)\epsilon_1$ and set $\epsilon_2 = 1 - K\epsilon_1 < \epsilon_1$. For small γ , this

guarantees that A_+ is positive (the rows in (3.2) are approximately $(1/3)A_{\text{row}(n)}$). Define $z_1 = -1 + x_1$ and $z_2 = K + x_2$. A computation shows

$$\begin{aligned} \det(M) &= \epsilon_1(z_2 - x_2) - \epsilon_2(z_1 - x_1) \\ &= \epsilon_1(K) - \epsilon_2(-1) = 1. \end{aligned}$$

Therefore $M \in \text{SL}(3, \mathcal{U})$, and W^+ gives a conjugacy of A^+ to A'' as required.

Let 0_i denote the $i \times i$ zero matrix. At the inductive step, we begin with a conjugacy of a positive matrix $(A^+)_{\ell-1}$ to a matrix with block form $\begin{pmatrix} C_{\ell-1} & 0 \\ 0 & 0 \end{pmatrix} = C_{\ell-1} \oplus 0_{\ell-1}$, and a splitting of $C_{\ell-1}$ to C_ℓ . The argument of the basic step produces a row splitting of $(A^+)_{\ell-1}$ to a positive matrix $(A^+)_\ell$ over \mathcal{U} and a conjugacy over \mathcal{U} of $(A^+)_\ell$ to $(C_\ell \oplus 0_{\ell-1}) \oplus 0_1$, which equals $C_\ell \oplus 0_\ell$.

The final claim of the lemma is clear by passing to transpose matrices. \square

Remark 3.3. If the nondiscrete ring \mathcal{U} is assumed to have a nontrivial unit, then in Lemma 3.2, the matrix A^+ can be chosen to have size equal to the matrix C (the extra zero blocks can be avoided). For this, in the proof at the stage of splitting the row $s' + t'$ of C , pick a, b from \mathcal{U} such that a closely approximates $1/2$ and b is a sufficiently small unit. In place of the matrices U, X'', M in the proof use

$$U = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad X'' = \begin{pmatrix} * \\ s' \\ t' \end{pmatrix}, \quad M = \begin{pmatrix} a & a-b \\ 1-a & b+(1-a) \end{pmatrix}.$$

Then $\det M = b$, so M is invertible over \mathcal{U} , and if a is chosen close to $1/2$ and b is sufficiently small, the positivity constraints will be satisfied.

Remark 3.4 (Matrices, module structures and splitting). Suppose \mathcal{U} is a unital ring, and let $\mathcal{U}[t]$ denote the ring of polynomials with coefficients in \mathcal{U} . If A is an $n \times n$ matrix A over \mathcal{U} , then the free \mathcal{U} module \mathcal{U}^n of row vectors is a right $\mathcal{U}[t]$ module \mathcal{M}_A , where the action of t is by $v \mapsto vA$. Two $n \times n$ matrices over \mathcal{U} are conjugate over \mathcal{U} if and only if their $\mathcal{U}[t]$ modules are isomorphic.

If a matrix C' is obtained by an elementary row splitting from a matrix C , with associated subdivision matrix U , then there is an embedding of \mathcal{M}_C into $\mathcal{M}_{C'}$, given by the rule $v \mapsto vU$. The conjugacy given by W^+ in Lemma 3.2 is constructed to extend the conjugacy of embedded copies of \mathcal{M}_A and \mathcal{M}_C obtained by lifting W . One finds an A^+ and W^+ by requiring further conditions on the vectors e_n, e_{n+1}, e_{n+2} .

The module viewpoint may give arguments which are easier and more conceptual, or help one find implementing matrices. On the other hand, it can be useful to have matrix arguments which can be verified by direct matrix computation.

It is worth noting that with any string of row splittings from a matrix C to a matrix C' , the module \mathcal{M}_C embeds as a $\mathcal{U}[t]$ submodule of $\mathcal{M}_{C'}$, and such that as a free \mathcal{U} module (forgetting the t action) $\mathcal{M}_{C'}$ is the internal direct sum of the embedded copy of \mathcal{M}_C and another free \mathcal{U} module.

We will also use the following easy lemma.

Lemma 3.5. *Let \mathcal{U} be a unital nondiscrete subring of \mathbb{R} . Suppose C is a positive square matrix over \mathcal{U} and M is a square matrix over \mathcal{U} of the (block) form $\begin{pmatrix} C & 0 \\ X & 0 \end{pmatrix}$ or $\begin{pmatrix} C & X \\ 0 & 0 \end{pmatrix}$. Then there is a positive matrix over \mathcal{U} which is conjugate over \mathcal{U} to M and which is SSE over \mathcal{U}_+ to C .*

Proof. Clearly it is sufficient to prove the lemma assuming $M = \begin{pmatrix} C & 0 \\ X & 0 \end{pmatrix}$. Pick $\kappa > 0$ in \mathcal{U} such that $X' := X + \kappa JC$ is positive, where J denotes a matrix of appropriate size with every entry equal to 1, and set $M' = \begin{pmatrix} C & 0 \\ X' & 0 \end{pmatrix}$. Then M' is SSE over \mathcal{U}_+ to C , and also conjugate over \mathcal{U} to M , since

$$M' = \begin{pmatrix} C & 0 \\ X' & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \kappa J & I \end{pmatrix} \begin{pmatrix} C & 0 \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\kappa J & I \end{pmatrix}.$$

Given ϵ in \mathcal{U} , define another matrix conjugate over \mathcal{U} to M ,

$$\begin{aligned} M_\epsilon &:= \begin{pmatrix} I & -\epsilon J \\ 0 & I \end{pmatrix} \begin{pmatrix} C & 0 \\ X' & 0 \end{pmatrix} \begin{pmatrix} I & \epsilon J \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} C - \epsilon JX' & 0 \\ X' & 0 \end{pmatrix} \begin{pmatrix} I & \epsilon J \\ 0 & I \end{pmatrix} = \begin{pmatrix} C - \epsilon JX' & (C - \epsilon JX')\epsilon J \\ X' & X'(\epsilon J) \end{pmatrix}. \end{aligned}$$

Fix $\epsilon > 0$ in \mathcal{U} sufficiently small to guarantee $C - \epsilon JX'$ is positive. Then M_ϵ is a positive matrix SSE over \mathcal{U}_+ to M' , and hence to C . \square

We are now prepared to prove the main result of this section.

Proof of Theorem 3.1. By assumption, for some ℓ we have matrices $A = A_{(0)}, A_{(1)}, \dots, A_{(\ell)} = B$ and for $0 \leq k < \ell$ an ESSE over \mathcal{U} ,

$$(3.3) \quad A_{(i)} = R_{(i)}S_{(i)} \quad , \quad A_{(i+1)} = S_{(i)}R_{(i)}.$$

By Proposition 2.9, we may assume all the matrices $A_{(i)}, R_{(i)}, S_{(i)}$ are nondegenerate. For each i , we can then by Proposition 2.10 associate to the ESSE (3.3) a diagram of splittings and a diagonal refactorization

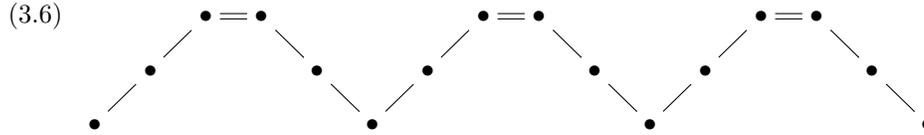
$$(3.4) \quad \begin{array}{ccc} & X_i & \xrightarrow{(D_i, D_i^{-1}X_i)} Y_i \\ & \swarrow & \searrow \\ A_{(i-1)} & & A_{(i)} \end{array}$$

as described in (2.1). By Lemma 2.13 we can lift each diagonal refactorization by a row and a column splitting to nondegenerate matrices conjugate over \mathcal{U} , giving a diagram of three levels,

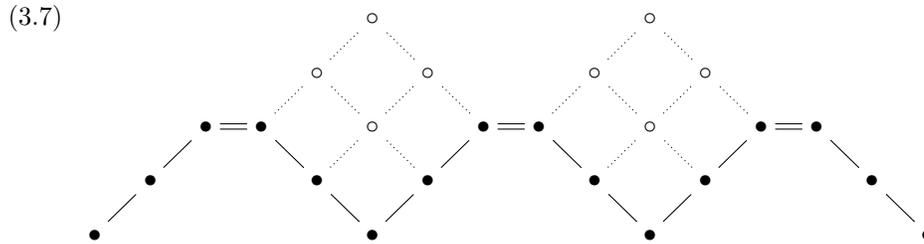
$$(3.5) \quad \begin{array}{ccc} & \bullet = \bullet & \\ & \swarrow \quad \searrow & \\ \bullet & & \bullet \\ \swarrow \quad \searrow & & \swarrow \quad \searrow \\ A_{(i-1)} & & A_{(i)} \end{array}$$

For visual clarity, we use the horizontal “=” in diagrams to indicate conjugacy over \mathcal{U} (not equality).

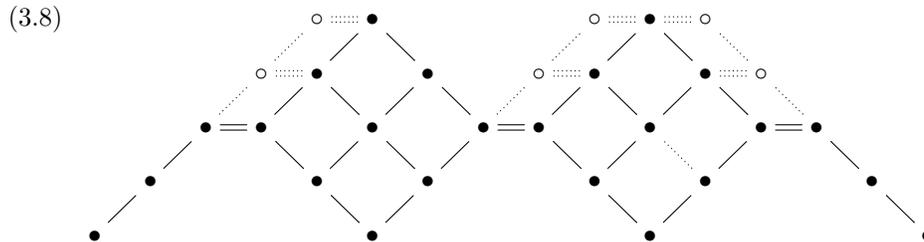
We consider an initial diagram (of three levels) formed by taking the union of the ℓ diagrams above (one for each ESSE). Arrows point southwest for row amalgamations and southeast for column amalgamations. For visual clarity, we suppress matrix names and arrow tips. Here is the initial diagram for the case $\ell = 3$.



We apply the Fiber Lemma 2.12 to construct a nondegenerate common column/row splitting for each pair of matrices in the diagram with a common row/column amalgamation, and iterate this move as far as possible. For our case $\ell = 3$, this produces the next diagram (with open circles and dotted lines reflecting additions to the diagram at this step).

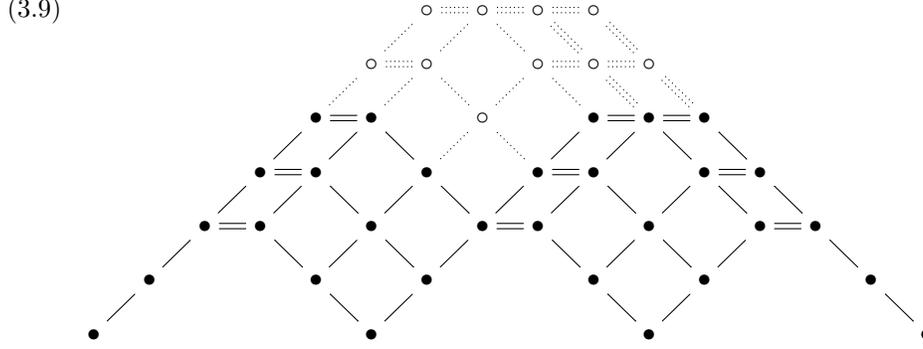


Next, we apply part (1) of the Splitting Lemma 3.2 to lift conjugacies of nondegenerate matrices by row or column splittings. Where there is a choice, for definiteness (only) we choose to lift by row splittings. For $\ell = 3$ this produces the following diagram. A nonhorizontal arrow here arising from the Splitting Lemma represents the composition of several splittings through nondegenerate matrices.



We iterate the application of Splitting Lemma and Fiber Lemma until we arrive at a final diagram of $2\ell + 1$ levels whose top level consists of matrices which are all conjugate. For $\ell = 3$, this happens at the next stage, and produces the diagram

(3.9) below.



At the top of the left side of the final diagram is a nondegenerate matrix C which is obtained from A by a sequence of row splittings through nondegenerate matrices over \mathcal{U} . By Part (2) of the Splitting Lemma 3.2, there is a matrix A^+ SSE over \mathcal{U}_+ to A and conjugate over \mathcal{U} to a matrix of the form $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$. Similarly, if E is the matrix on the right side of the top level of the final diagram, then there is a positive matrix B^+ which is SSE over \mathcal{U}_+ and conjugate over \mathcal{U} to a matrix of the form $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$. If A^+ and B^+ are not of the same size, then we may apply Lemma 3.5 to enlarge one of them, and assume they have the same size. Because C and E are conjugate over \mathcal{U} , it then follows that A^+ and B^+ are conjugate over \mathcal{U} .

This concludes the proof, apart from the “moreover” claim for the conjugating matrix. This is perhaps already clear from previous work, but we will give a self contained proof in the following lemma. \square

Lemma 3.6. *Suppose \mathcal{U} is a nondiscrete unital subring of \mathbb{R} , and A, B are $n \times n$ positive matrices conjugate over \mathcal{U} . Then there are positive matrices A', B' SSE over \mathcal{U}_+ to A, B respectively and a matrix U invertible over \mathcal{U} such that $U^{-1}A'U = B'$, $\det U > 0$, and U sends positive eigenvectors to positive eigenvectors.*

Proof. We are given $U \in \text{GL}(n, \mathcal{U})$ such that $U^{-1}AU = B$. We may assume (if necessary after replacing U with $-U$) that U sends positive eigenvectors to positive eigenvectors. In the case $\det U < 0$, it would suffice to have some W invertible over \mathcal{U} such that $AW = WA$, $\det W < 0$ and W respects positive eigenvectors of A . (We could then replace U with WU .)

If no such W exists, then for some small $\epsilon > 0$ we will define an $(n + 1) \times (n + 1)$ matrix A' as a row splitting of A , by splitting the first row A_1 of A as $\epsilon A_1 + (1 - \epsilon)A_1$. Here $\epsilon > 0$, $\epsilon \in \mathcal{U}$ and ϵ is small enough that $A' > 0$. Let $E_{ij}(s)$ denote the $n + 1 \times n + 1$ matrix equal to I except that the ij entry is s . Let $F = E_{12}(-\epsilon)E_{21}(1)$. Then

$$FA'F^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} := A'' .$$

Because the matrix $K = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$ has negative determinant, commutes with A'' and fixes eigenvectors for nonzero eigenvalues, the matrix $W = F^{-1}KF$ will have the same properties with respect to $A' = F^{-1}A''F$.

Now define B' from B in the same way. The matrices A', B' are positive, SSE over \mathcal{U}_+ to A, B respectively, and conjugate by a matrix with positive determinant which sends positive eigenvectors to positive eigenvectors. \square

We prepare for the last result of this section with the next lemma.

Lemma 3.7. *Suppose A, B, D, B' are matrices over a field \mathcal{U} such that D is diagonal nonsingular, $A = D^{-1}BD$ and B' is a row splitting of B . Then there is a row splitting A' of A and a diagonal matrix D' over \mathcal{U} such that $A' = (D')^{-1}BD'$.*

The same statement is true if “row” is replaced with “column”.

Proof. Clearly it suffices to prove the row statement. Let U be the subdivision matrix for the assumed row splitting, $BU = UB'$. Define the diagonal matrix D' in the obvious way, $D'(i, i) = D(\bar{i}, \bar{i})$ where \bar{i} satisfies $U(\bar{i}, i) = 1$. Define A' to be $(D')^{-1}BD'$. Then U is also the subdivision matrix for a row splitting of A to A' . For an example of this, take

$$A = \begin{pmatrix} a & b\frac{\delta_1}{\delta_2} \\ c\frac{\delta_2}{\delta_1} & d \end{pmatrix} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad D = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

$$A' = \begin{pmatrix} a_1 & a_1 & \frac{\delta_1}{\delta_2}b_1 \\ a_2 & a_2 & \frac{\delta_1}{\delta_2}b_2 \\ \frac{\delta_2}{\delta_1}c & \frac{\delta_2}{\delta_1}c & d \end{pmatrix} \quad B' = \begin{pmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ c & c & d \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

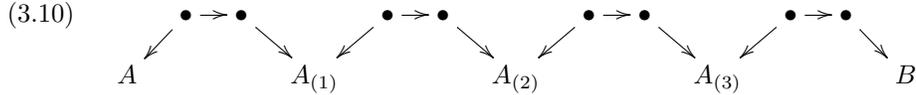
□

If \mathcal{U} is not a field, then the definition of A' in the proof of Lemma 3.7 above might not give a matrix over \mathcal{U} .

We will need the following result from [15].

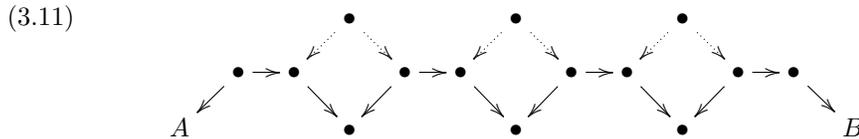
Theorem 3.8. *Suppose A, B are nondegenerate matrices SSE over \mathcal{U}_+ , where \mathcal{U} is a subfield of \mathbb{R} . Then there is a nondegenerate matrix C over \mathcal{U}_+ and a nonsingular diagonal matrix D over \mathcal{U}_+ such that C is reached from A by finitely many row splittings through matrices over \mathcal{U}_+ and $D^{-1}CD$ is reached from B by finitely many column splittings through matrices over \mathcal{U}_+ . If A is primitive, then C is primitive.*

Proof. The proof is a simplification of part of the proof of Theorem 3.1. We begin with a lag ℓ SSE from A to B through nondegenerate matrices over \mathcal{U}_+ . As in (3.4), from each elementary SSE we produce a row splitting, diagonal refactorization and column splitting. For example, with $\ell = 4$ we get a diagram

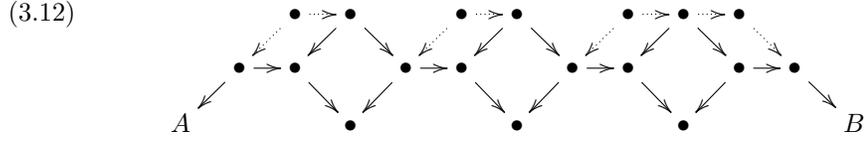


A southwest pointing arrow represents a row amalgamation. A southeast pointing arrow represents a column amalgamation. A horizontal arrow represents a diagonal refactorization over \mathcal{U} .

Above each $A_{(i)}$, we apply Lemma 2.12 to produce a common splitting: With this move, we have added another level to the top of the diagram:

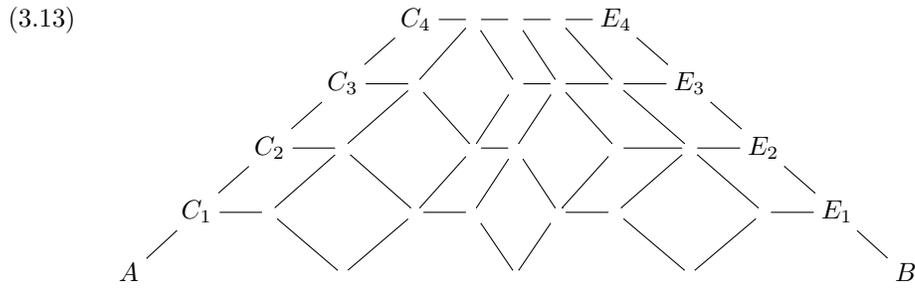


Then we apply Lemma 3.7 to lift each diagonal refactorization on the old top row by a splitting to the new top row:



When there is a choice, for definiteness (only) we make the choice to lift with a row splitting.

Iterating this pair of moves $\ell - 1$ additional times, we produce a diagram with $\ell + 1$ horizontal levels. For $\ell = 4$, this is the following diagram, in which we insert some matrix names (whose generalizations to arbitrary ℓ should be clear) and for visual simplicity suppress arrowheads and bullets.



Define $A' = C_\ell$ and $B' = E_\ell$. The diagonal matrix D such that $D^{-1}A'D = B'$ is produced by composing the ℓ diagonal refactorizations on the top level of the diagram. If A is primitive, then C is primitive, because C is nondegenerate and SSE over \mathbb{R}_+ to A . \square

4. THE CENTRALIZER

Definition 4.1. Given an $n \times n$ matrix A over \mathbb{R} , we let $\text{Cent}_{\mathbb{R}}(A)$ denote $\{B \in \text{GL}(n, \mathbb{R}) : AB = BA\}$, the centralizer of A in $\text{GL}(n, \mathbb{R})$. Let $\text{GL}_+(n, \mathbb{R})$ denote the connected component of the identity in $\text{GL}(n, \mathbb{R})$ (the matrices with positive determinant). Define $\text{Cent}_{\mathbb{R}}^+(A)$ to be $\text{Cent}_{\mathbb{R}}(A) \cap \text{GL}_+(n, \mathbb{R})$,

This section provides some background and notation for $\text{Cent}_{\mathbb{R}}(A)$, needed for the results to come on strong shift equivalence over nondiscrete unital subrings of \mathbb{R} .

In the next lemma, \mathcal{U} could be for example a field, or it could be obtained from a real algebraic number ring by inverting all but finitely many primes.

Lemma 4.2 (Centralizer Lemma). *Suppose that \mathcal{U} is a dense unital subring of \mathbb{R} which contains an ideal $J \neq \mathcal{U}$ such that every element of \mathcal{U} outside J is a unit in \mathcal{U} . Suppose that A is an $n \times n$ matrix over \mathcal{U} .*

Then every connected component of $\text{Cent}_{\mathbb{R}}(A)$ contains an element of $\text{GL}(n, \mathcal{U})$.

Proof. Let \mathbb{F} denote the field of fractions of \mathcal{U} . The set of (not necessarily invertible) real matrices which commute with A , as the solution set of $AX - XA = 0$, is a real vector space $V_{\mathbb{R}}$ in which the matrices over \mathbb{F} are an \mathbb{F} vector space $V_{\mathbb{F}}$ of equal dimension. Thus $V_{\mathbb{F}}$ is dense in $V_{\mathbb{R}}$. It suffices to show $\text{GL}(n, \mathcal{U}) \cap V_{\mathbb{R}}$ is dense in

$V_{\mathbb{F}}$ (in which case it is also dense in every connected component of $\mathrm{GL}(n, \mathbb{R}) \cap V_{\mathbb{R}}$, which is $\mathrm{Cent}_{\mathbb{R}}(A)$).

If $J = \{0\}$, then \mathcal{U} is a field. Then any element of $V_{\mathbb{F}}$ close to an element of $\mathrm{GL}(n, \mathbb{R}) \cap V_{\mathbb{R}}$ has nonzero determinant and thus lies in $\mathrm{GL}(n, \mathcal{U})$.

So, let J be a proper ideal of \mathcal{U} . Then J is dense in \mathbb{R} . Suppose $X \in V_{\mathbb{F}}$, and $\epsilon > 0$. Pick a nonzero $d_1 \in \mathcal{U}$ such that $d_1 X$ has all entries in \mathcal{U} . Pick $d_2 \in \mathcal{U}$ such that $|d_1 d_2 - 1| < \epsilon$. Let $Y = d_1 d_2 X$. Then Y has entries in \mathcal{U} and $\|Y - X\| < \epsilon \|X\|$. Pick c in J such that $|1 - c| < \epsilon$. Let $M = (1 - c)I + cY$. Then $M \in V_{\mathbb{F}}$ and

$$\|Y - M\| = \|(1 - c)(Y - I)\| \leq \epsilon \|Y - I\| .$$

Since $\det(M) = \det(I + c(Y - I)) \equiv 1 \pmod{J}$, we also have $M \in \mathrm{GL}(n, \mathcal{U})$. \square

For completeness we recall an example from [16].

Example 4.3. Let $U = \mathbb{Z}[1/p]$, where p is a prime (e.g., 5) which does not split in the algebraic number ring $\mathbb{Z}[\sqrt{3}]$. Let A be a nonzero 2×2 matrix of the form $c_1 I + c_2 B$, with c_1, c_2 in \mathcal{U} , where $B = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$.

Then $\mathrm{GL}(2, \mathcal{U})$ does not intersect every connected component of $\mathrm{Cent}_{\mathbb{R}}(A)$. If C is the 3×3 matrix $B \oplus 1$, then $\mathrm{GL}(3, \mathcal{U})$ does not intersect every connected component of $\mathrm{Cent}_{\mathbb{R}}^+(C)$.

Proof. There is an isomorphism of fields from $\mathbb{Q}[\sqrt{3}]$ to $\mathbb{Q}[A]$ induced by $\sqrt{3} \mapsto B$. A fundamental unit for the algebraic number ring $\mathbb{Z}[\sqrt{3}]$ is $2 + \sqrt{3}$, which has positive norm 1. If p is an odd prime and $p > 3$ and 3 is not a square mod p (for example $p = 5$), then p does not split in $\mathbb{Z}[\sqrt{3}]$ [21, p.74].

The matrix A has distinct real eigenvalues and can be diagonalized over the reals. When diagonalized, its centralizer becomes all diagonal matrices and is all linear combinations over \mathcal{U} of I, A . So that is also true when it is not diagonalized. The real centralizer will be a direct sum of two copies of the reals and has four components. Some components have negative determinant. The centralizer of A within the 2×2 matrices over U consists of linear combinations of I, A over U , and is isomorphic as a ring to the quadratic number ring $R = \mathbb{Z}[\sqrt{3}]$ with the prime p inverted. By assumption p is prime in R , so any unit of \mathcal{U} has the form $p^m(2 + \sqrt{3})^n$ for some integers m, n . The norms of all units in \mathcal{U} are positive, which translates to the determinants of all elements of $\mathrm{GL}(2, \mathcal{U})$ in the centralizer of A being positive. Therefore the negative determinant components of $\mathrm{Cent}_{\mathbb{R}}(A)$ will not intersect $\mathrm{GL}(2, \mathcal{U})$.

A matrix in the centralizer of C will act like a matrix in the centralizer of A together with multiplication by a scalar on the fixed direction. A negative-determinant component of $\mathrm{Cent}_{\mathbb{R}}(A)$, with multiplication by a negative number on the fixed direction, will yield a positive-determinant component of $\mathrm{Cent}_{\mathbb{R}}(C)$ which does not intersect $\mathrm{GL}(3, \mathcal{U})$. \square

Definition 4.4. Given a square real matrix A , let $\mathcal{J}(A)$ denote the set of pairs (λ, j) such that $\lambda \in \mathbb{R}$, $j \in \mathbb{N}$ and the Jordan form of A contains a $j \times j$ Jordan block for λ . Then define

$$\gamma(A) = |\mathcal{J}(A)| .$$

For A $n \times n$ over \mathbb{R} and $(\lambda, j) \in \mathcal{J}(A)$, define the vector space

$$V(A, \lambda, j) = \{x(A - \lambda I)^{j-1} \in \ker(A) : x \in \mathbb{R}^n \setminus \mathrm{image}(A)\} \cup \{0\} .$$

A matrix B in $\text{Cent}_{\mathbb{R}}(A)$ maps each $V(A, \lambda, j)$ to itself. Let $\sigma(\lambda, j)$ be the sign of the determinant of this map determined by B .

Recall, $\pi_0(X)$ is the set of connected components of a topological space X .

Proposition 4.5. *Let A be an $n \times n$ real matrix. Then with γ as defined above,*

$$|\pi_0(\text{Cent}_{\mathbb{R}}(A))| = 2^{\gamma(A)} .$$

Two matrices lie in the same component if and only if they have the same sign $\sigma(\lambda, j)$ for all (λ, j) in $\mathcal{J}(A)$.

Proof. If A is zero, the claim holds because $|\pi_0(\text{GL}(n, \mathbb{R}))| = 2$. Now suppose $A \neq 0$. A is a sum of commuting nonzero real matrices A_λ , where λ denotes a root of χ_A with nonnegative imaginary part, and $A_\lambda - \lambda I$ is nilpotent if $\lambda \in \mathbb{R}$, and $(A_\lambda - \lambda I)(A_\lambda - \bar{\lambda}I)$ is nilpotent if $\lambda \in \mathbb{R} \setminus \mathbb{C}$. The centralizer $\text{Cent}_{\mathbb{R}}(A)$ is homeomorphic to the product $\prod_{\lambda} \text{Cent}_{\mathbb{R}}(A_\lambda)$. So it suffices to show the claim for each A_λ .

If λ is real, then $\text{Cent}_{\mathbb{R}}(A_\lambda) = \text{Cent}_{\mathbb{R}}(A_\lambda - \lambda I)$, and we can consider $\text{Cent}_{\mathbb{R}}(M)$ for M nilpotent. Let C_j denote a matrix of the form $J \otimes I$, where J is a $j \times j$ Jordan block with zero diagonal. Then $\text{Cent}_{\mathbb{R}}(C_j)$ has a banded form, e.g. for C_3 a $(3k \times 3k) \times (3k \times 3k)$ matrix,

$$C_3 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Cent}_{\mathbb{R}}(C_3) = \left\{ \begin{pmatrix} X & Y & Z \\ 0 & X & Y \\ 0 & 0 & X \end{pmatrix} : X \in \text{GL}(k, \mathbb{R}) \right\} .$$

Each $\text{Cent}_{\mathbb{R}}(C_j)$ has exactly two connected components, depending on the sign of the determinant of the repeated diagonal block, which is $\sigma(\lambda, j)$. Up to conjugacy, the nilpotent matrix M will be block diagonal of the form $\text{diag}(M_1, M_2, \dots, M_j)$, i.e.

$$M = \begin{pmatrix} M_1 & 0 & 0 & \dots & 0 \\ 0 & M_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & M_j \end{pmatrix}$$

where $M_i = C_{n(i)}$, $n(1) > n(2) > \dots > n(j)$ and $j = \gamma(M)$. $\text{Cent}_{\mathbb{R}}(M)$ is a subset of the set of block upper triangular matrices such that an element of $\text{Cent}_{\mathbb{R}}(M_i)$ occupies the i th diagonal block. There is a homotopy from $\text{Cent}_{\mathbb{R}}(M)$ to the set of its block diagonal matrices, which is homeomorphic to $\prod_j \text{Cent}_{\mathbb{R}}(M_j)$, which has 2^j connected components.

If λ is complex and A_λ is $(2k) \times (2k)$, then let M be the $k \times k$ complex matrix which is the direct sum of the λ -Jordan blocks in the Jordan form of A_λ (or A). Then $\text{Cent}_{\mathbb{R}}(A_\lambda)$ is homeomorphic to $\text{Cent}_{\mathbb{C}}(M)$, the centralizer of M in $\text{GL}(k, \mathbb{C})$. The triangular structure described earlier applies to $\text{Cent}_{\mathbb{C}}(M)$. However, because $\text{GL}(n, \mathbb{C})$ is connected for every n , we have that $\text{Cent}_{\mathbb{C}}(M)$ is connected. \square

5. FROM PATHS OF SIMILAR MATRICES TO STRONG SHIFT EQUIVALENCE

In this section, we will see how to pass from a path of positive conjugate matrices to a strong shift equivalence through positive matrices. (The problem of finding such a path we consider later.) For completeness, we begin with a proof for the path lifting Proposition 5.3, for which we make some preparation. Below, the particular choice of norm for \mathbb{R}^n is unimportant. The next lemma was proved in [15] with a citation to [9]; we include a proof for completeness.

Lemma 5.1. *Suppose B is a matrix over \mathbb{R} and $\epsilon > 0$. Then there exists $\delta > 0$ such that if $\|B - B'\| < \delta$ and B' is conjugate over \mathbb{R} to B , then there exists U in $\text{GL}(n, \mathbb{R})$ such that $U^{-1}BU = B'$ and $\|U - I\| < \epsilon$.*

Proof. We begin with a Claim: Suppose $\epsilon > 0$ and M is an $n \times n$ matrix of rank r over scalar field \mathbb{C} or \mathbb{R} , and u_1, \dots, u_{n-r} is a basis of $\ker(M)$. Then there is $\delta > 0$ such that for any M' with $\|M - M'\| < \delta$ and $\text{rank}(M') = r$, there is a basis u'_1, \dots, u'_{n-r} of $\ker(M')$ with $\|u_j - u'_j\| < \epsilon\|u_j\|$, for each j .

Proof of Claim. Without loss of generality, suppose $0 < r < n$. To set notation, we use row vectors for $\ker M = \{v : vM = 0\}$. Let proj_W denote orthogonal projection onto W . Let $\text{col}M$ denote the vector space generated by column vectors of M .

Within the set of $n \times n$ matrices M of rank r , the map $\text{proj}_{\text{col}M}$ varies continuously with M . (For M' near a given M , the same r linearly independent columns can be used to construct an orthonormal basis with the Gram-Schmidt algorithm.) So we may suppose δ is small enough that $\|\text{proj}_{\text{col}M'}(v)\| \leq \|\text{proj}_{\text{col}M}(v)\| + \epsilon\|v\|$, for all v . Set $u'_j = \text{proj}_{\ker M'}(u_j)$. Considering $(\ker M')^\perp = \{v^{tr} : v \in \text{col}M'\}$, we have

$$\begin{aligned} \|u_j - u'_j\| &= \|\text{proj}_{(\ker M')^\perp}(u_j)\| = \|\text{proj}_{\text{col}M'}(u_j^{tr})\| \\ &\leq \|\text{proj}_{\text{col}M}(u_j^{tr})\| + \epsilon\|u_j^{tr}\| = \epsilon\|u_j^{tr}\|. \end{aligned}$$

This proves the claim.

Now suppose λ is an eigenvalue of B and $\mathcal{J}_\lambda = \{u_1, \dots, u_s\}$ is a Jordan basis for the restriction of $B - \lambda I$ to $\ker(B - \lambda I)^n$. ($\cup_\lambda \mathcal{J}_\lambda$ is a Jordan basis for B .) Define

$$\mathcal{J}_\lambda(t) = \{u_i \in \mathcal{J}_\lambda : u_i(B - \lambda I)^t = 0 \text{ and } u_i \notin \text{image}(B - \lambda I)\}.$$

(The number of vectors in $\mathcal{J}_\lambda(t)$ equals the number of $t \times t$ Jordan λ -blocks in the Jordan form of B .) For B' close to B , let $\{u'_1, \dots, u'_s\}$ be the nearby basis of $\ker(B' - \lambda I)^n$, given by the Claim. With B' close enough to B ,

$$u_i \in \mathcal{J}_\lambda(t) \implies u'_i(B' - \lambda I)^{t-1} \neq 0.$$

Consider the t in decreasing order, we then deduce from the conjugacy of B and B' that also

$$u_i \in \mathcal{J}_\lambda(t) \implies u'_i(B' - \lambda I)^t = 0.$$

For λ real, those vectors u_i, u'_i can be chosen to be real, and the map on \mathcal{J}_λ defined by

$$u_i(B - \lambda I)^j \mapsto u'_i(B' - \lambda I)^j, \quad \text{if } u_i \in \mathcal{J}_\lambda(t) \text{ and } 0 \leq j < t,$$

determines a map $\ker(B - \lambda I)^n \rightarrow \ker(B' - \lambda I)^n$ which conjugates the restrictions of B and B' to these invariant subspaces. For λ not real, say with positive imaginary part, pull back the complex conjugacy to define a map from a real Jordan form basis for B for $\ker(B - \lambda I)^n(B - \bar{\lambda}I)^n$ to a corresponding nearby real Jordan basis for B' for $\ker(B' - \lambda I)^n(B' - \bar{\lambda}I)^n$.

The matrix U in $\text{GL}(n, \mathbb{R})$ implementing these maps on invariant subspaces induces a conjugacy of B and B' and is close to the identity. \square

Below, we suppose A is an $n \times n$ real matrix, $\text{Cent}(A) = \{U \in \text{GL}(n, \mathbb{R}) : UA = AU\}$; $\text{Conj}(A) = \{U^{-1}AU : U \in \text{GL}(n, \mathbb{R})\}$; $\gamma : U \mapsto U^{-1}AU$; the topology of $\text{Conj}(A)$ is by the metric induced by a matrix norm, and the image of π has the quotient topology.

Proposition 5.2. *The map ϕ which makes the following diagram commute*

$$\begin{array}{ccc}
 \mathrm{GL}(n, \mathbb{R}) & & \\
 \downarrow \gamma & \searrow \pi & \\
 & & \mathrm{GL}(n, \mathbb{R})/\mathrm{Cent}(A) \\
 & \swarrow \phi & \\
 \mathrm{Conj}(A) & &
 \end{array}$$

is a homeomorphism.

Proof. For U, V in $\mathrm{GL}(n, \mathbb{R})$, we have $\gamma(U) = \gamma(V)$ if and only if $VU^{-1} \in \mathrm{Cent}(A)$. So, ϕ is a well defined bijection. The map ϕ is continuous because γ is continuous, π is open and $\mathrm{GL}(n, \mathbb{R})/\mathrm{Cent}(A)$ has the quotient topology.

It remains to show that ϕ is an open map. This holds if γ is an open map. Suppose \mathcal{V} is an open subset of $\mathrm{GL}(n, \mathbb{R})$ and $V \in \mathcal{V}$. Choose $\epsilon > 0$ such that \mathcal{V} contains the open set $\{VU \in \mathrm{GL}(n, \mathbb{R}) : \|U - I\| < \epsilon\}$. Then $\gamma(\mathcal{V})$ contains $\{U^{-1}CU : \|U - I\| < \epsilon\}$, which by Lemma 5.1 contains some neighborhood of C in $\mathrm{Conj}(C) = \mathrm{Conj}(A)$. This shows the map γ is open and finishes the proof. \square

Above, $((\mathrm{GL}(n, \mathbb{R}), \pi, \mathrm{GL}(n, \mathbb{R})/\mathrm{Cent}(A))$ is a principal bundle [11, Ch. 4.2]. The projection π is locally trivial: for every x in $\mathrm{GL}(n, \mathbb{R})$, there is a neighborhood \mathcal{U} of x and a neighborhood \mathcal{V} of πx and a homeomorphism $h : \mathcal{U} \rightarrow \mathcal{V} \times \mathrm{Cent}(A)$ such that on \mathcal{U} , π is equal to h followed by projection onto \mathcal{V} , $(v, c) \mapsto v$.

Proposition 5.3. *Suppose $(A_t)_{0 \leq t \leq 1}$ is a path of conjugate $n \times n$ real matrices, $U \in \mathrm{GL}(n, \mathbb{R})$ and $U^{-1}A_0U = A_0$. Then there is a path (G_t) in $\mathrm{GL}(n, \mathbb{R})$ such that $G_0 = U$ and $G_t^{-1}A_0G_t = A_t$ for $0 \leq t \leq 1$.*

Proof. The map γ has the topological properties of the principal bundle projection π in Proposition 5.2. The proposition translates to γ the path lifting property which π enjoys on account of its local triviality as a projection. \square

Let $\mathcal{H}(I, \mathrm{Cent}_{\mathbb{R}}^+(A))$ denote the homotopy classes of paths in $\mathrm{GL}_+(n, \mathbb{R})$ from the identity to $\mathrm{Cent}_{\mathbb{R}}^+(A)$ (by homotopy through paths with initial point I and terminal points in $\mathrm{Cent}_{\mathbb{R}}^+(A)$). In a topological space X , $\pi_0(X)$ denotes the set of connected components and $\pi_1(x, X)$ denotes the fundamental group at basepoint x in X .

Proposition 5.4. *Suppose $(A_t)_{0 \leq t \leq 1}$ is a loop from A to A in $\mathrm{Conj}(A)$. Then there is a path (G_t) in $\mathrm{SL}(n, \mathbb{R})$ such that $G_0 = I$ and $A_t = (G_t)^{-1}AG_t$, $0 \leq t \leq 1$. Moreover, the homotopy class of the loop (A_t) determines both the element of $\mathcal{H}(I, \mathrm{Cent}_{\mathbb{R}}^+(A))$ containing (G_t) and the connected component of $\mathrm{Cent}_{\mathbb{R}}^+(A)$ which contains G_1 . The induced maps*

$$\begin{aligned}
 \pi_1(A, \mathrm{Conj}(A)) &\rightarrow \mathcal{H}(I, \mathrm{Cent}_{\mathbb{R}}^+(A)) \quad , \\
 \pi_1(A, \mathrm{Conj}(A)) &\rightarrow \pi_0(\mathrm{Cent}_{\mathbb{R}}^+(A))
 \end{aligned}$$

are bijections.

Proof. Proposition 5.3 explains the existence of the lift of (A_t) to a path (G_t) in $\mathrm{GL}(n, \mathbb{R})$ beginning at $G_0 = I$. By continuity, each G_t has positive determinant, and we may replace G_t with $(\det G_t)^{-1/n}G_t$ to put the conjugating path into $\mathrm{SL}(n, \mathbb{R})$. It is straightforward to check the remaining claims about well defined induced bijections. \square

Definition 5.5. Suppose \mathcal{U} is a nondiscrete unital subring of the reals. Given A an $n \times n$ matrix over \mathcal{U} . We say the *centralizer condition* holds for (A, \mathcal{U}) if every connected component of $\text{Cent}_{\mathbb{R}}^+(A)$ has nonempty intersection with $\text{GL}(n, \mathcal{U})$. We say that \mathcal{U} *satisfies the centralizer condition* if the centralizer condition holds for (A, \mathcal{U}) for every square matrix A over \mathcal{U} .

One equivalent statement of the Centralizer Condition is that $\text{Cent}_{\mathbb{R}}^+(A)$ is generated by $\mathcal{U} \cap \text{Cent}_{\mathbb{R}}^+(A)$ and the connected component of the identity in $\text{Cent}_{\mathbb{R}}^+(A)$.

Lemma 5.6. *Suppose A is a positive $n \times n$ real matrix. There is an $\epsilon > 0$ such that for $U \in \text{SL}(n, \mathbb{R})$ with $\|U - I\| < \epsilon$, U can be written as a product of $m = (n+4)(n-1)$ basic elementary matrices, $U = E_1 \cdots E_m$, where each E_k depends continuously on U .*

Proof. The given U can be made upper triangular by $(n-1) + (n-2) + \cdots + 1 = n(n-1)/2$ operations of adding multiples of rows successively to lower rows. For U close to the identity, at each stage the diagonal terms will remain positive, and the multiples of row i added to lower rows to zero out entries in column i will depend continuously on U . Likewise, the same number of additions of lower rows to upper rows will diagonalize U . Finally, for $a \neq 0$,

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} 1 & a(a-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

So, we can multiply a diagonal determinant 1 matrix by $4(n-1)$ elementary matrices to produce the identity. In total we have factored U as a product of $m = 2[n(n-1)/2] + 4(n-1) = (n+4)(n-1)$ basic elementary matrices. We may fix the order of operations. Then in each E_k , there is a single offdiagonal element which is allowed to be nonzero (or zero), and it varies continuously as a function of U . \square

Lemma 5.7. *Given $0 < \epsilon < \kappa$ and $n \in \mathbb{N}$, there is a $\delta > 0$ such that for every $n \times n$ real matrix M with all entries bounded below by ϵ and above by κ , for every $U \in \text{SL}(n, \mathbb{R})$ with $\|U - I\| < \delta$, the matrix $U^{-1}MU$ is positive and SSE over \mathbb{R}_+ to M .*

Proof. Pick $\delta > 0$ small enough that for every $n \times n$ positive matrix M with entries bounded below by ϵ and above by K , with $m = (n+4)(n-1)$ we have

- for $U \in \text{SL}(n, \mathbb{R})$ with $\|U - I\| < \delta$, there is a continuous factorization $U = E_1 \cdots E_m$ of U into basic elementary matrices, as in Lemma 5.6, and
- with $E_0 = I$ and $V_i = E_0 \cdots E_i$, for $0 \leq i < m$ all of the matrices $V_i^{-1}MV_i$, $V_i^{-1}MV_iE_{i+1}$ and $E_{i+1}^{-1}V_i^{-1}MV_i$ are positive.

Given such a matrix M , set $B_i = V_i^{-1}MV_i$; then $B_0 = M$ and $B_m = U^{-1}MU$. For $1 \leq i \leq m$, B_i is positive, and one of the pairs $(E_i, E_i^{-1}B_i)$, (E_iB_i, E_i^{-1}) will give an ESSE over \mathbb{R}_+ from B_{i-1} to B_i . Thus M and $U^{-1}MU$ are positive matrices which are SSE over \mathbb{R}_+ . \square

Lemma 5.8. *Suppose \mathcal{U} is a nondiscrete unital subring of \mathbb{R} , B is an $n \times n$ positive matrix over \mathcal{U} and $d > 0$.*

Then there is $\epsilon > 0$ such that $V \in \text{GL}(n, \mathcal{U})$ with $\|V - dI\| < \epsilon$ implies the matrix $V^{-1}BV$ is positive and is SSE over \mathcal{U}_+ to B .

Proof. If $\|V - dI\| < \epsilon$ with ϵ sufficiently small, then we may add small positive multiples of a row i of V to other rows to make all off diagonal entries of column

i positive and still small. Iterating, we may find nonnegative elementary matrices E_1, \dots, E_k over \mathcal{U} , with $k \leq n(n-1)$, such that with $E = E_k E_{k-1} \cdots E_1$, the matrix EV is a positive matrix in $\text{GL}(n, \mathcal{U})$.

Set $E_0 = I$ and $B_0 = B$. For $1 \leq i \leq k$ set $B_i = E_i B_{i-1} E_i^{-1}$. For ϵ small, we may choose the matrices E_i close enough to I that all of the following matrices are also positive: BE^{-1} ; $V^{-1}(BE^{-1})$; $B_{i-1}E^{-1}$ and $B_i = E_i B_{i-1} E_i^{-1}$, for $1 \leq i \leq k$.

Then $V^{-1}BV = (V^{-1}BE^{-1})(EV) > 0$, and the pair $(V^{-1}BE^{-1}, EV)$ gives an ESSE over \mathcal{U}_+ from $V^{-1}BV$ to EBE^{-1} . There is also an SSE over \mathcal{U}_+ between $B = B_0$ and $EBE^{-1} = B_k$: for $1 \leq i \leq k$, the pair $(B_{i-1}E_i^{-1}, E_i)$ gives an ESSE over \mathcal{U}_+ from B_{i-1} to B_i . □

Definition 5.9. Let \mathcal{U} be a semiring in \mathbb{R} . Matrices A, B are SSE over \mathcal{U}_+ , through positive $n \times n$ matrices, if for some $\ell \in \mathbb{N}$ there are $n \times n$ positive matrices $A = A_0, A_1, \dots, A_\ell = B$ such that A_{i-1} is ESSE over \mathcal{U}_+ to A_i , $1 \leq i \leq \ell$.

In the next theorem, part (1) was proved in [15]. Parts (2) and (3) were proved in [16] under the condition that elements of $\text{GL}(n, \mathcal{U})$ are dense in $\text{GL}(n, \mathbb{R})$. Here we remove this condition by working with the special linear group and scalar matrices.

Theorem 5.10 (Path Theorem). *Let \mathcal{U} be a unital nondiscrete subring of \mathbb{R} , and suppose $(A_t)_{0 \leq t \leq 1}$ is a path of positive, real $n \times n$ matrices, all in the same conjugacy class over \mathbb{R} , from $A = A_0$ to $B = A_1$. Then the following hold*

- (1) *A and B are SSE over \mathbb{R}_+ , through positive $n \times n$ matrices.*
- (2) *Suppose there is a path (G_t) in $\text{GL}(n, \mathbb{R})$ such that $G_0 = I$ and $G_t^{-1}AG_t = A_t$ for all t , and there is a W in $\text{GL}(n, \mathcal{U})$ such that $W^{-1}AW = B$ and WG_1^{-1} is in the connected component of the identity in $\text{Cent}_{\mathbb{R}}(A)$.
Then A and B are SSE over \mathcal{U}_+ , through positive $n \times n$ matrices.*
- (3) *Suppose A and B are conjugate matrices over \mathcal{U} such that every connected component of $\text{Cent}_{\mathbb{R}}(A)$ contains a matrix from $\text{GL}(n, \mathcal{U})$.
Then A and B are SSE over \mathcal{U}_+ , through positive $n \times n$ matrices.*
- (4) *Suppose A and B are conjugate matrices over \mathcal{U} and \mathcal{U} is a field, or more generally contains an ideal $J \neq \mathcal{U}$ such that every element of \mathcal{U} outside J is a unit in \mathcal{U} .
Then A and B are SSE over \mathcal{U}_+ , through positive $n \times n$ matrices.*

Proof. (1) Proposition 5.3 gives us a path (G_t) in $\text{SL}(n, \mathbb{R})$ with $G_0 = I$ and $(G_t^{-1}AG_t) = (A_t)$. The entries of the positive matrices A_t are by compactness of the path uniformly bounded below by some positive ϵ and above by some κ . Let $\delta > 0$ be chosen as in Lemma 5.7 for ϵ, κ . Now by uniform continuity pick $k \in \mathbb{N}$ such that $s \leq t \leq s + \frac{1}{k}$ implies $\|G_t(G_s^{-1}) - I\| < \delta$. By Lemma 5.7, $A_{i/k}$ is SSE over \mathbb{R}_+ to $A_{(i+1)/k}$, for $0 \leq i < k$. Therefore $A = A_0$ and $B = A_1$ are SSE over \mathbb{R}_+ .

(2) Let X_t , $0 \leq t \leq 1$, be a path from the identity to WG_1^{-1} in the centralizer of A in $\text{GL}(n, \mathbb{R})$. Then $(X_t G_t)$ is a path from the identity to W in $\text{GL}(n, \mathbb{R})$, and $W^{-1}AW = B$. For all t , $(X_t G_t)^{-1}A(X_t G_t) = G_t^{-1}AG_t$. So we may assume $G_1 \in \text{GL}(n, \mathcal{U})$.

Next define the path H_t in $\text{SL}(n, \mathbb{R})$ by $H_t = c_t G_t$, $0 \leq t \leq 1$, where $c_t = (\det(G_t))^{-1/n}$. We have $H_t^{-1}AH_t = G_t^{-1}AG_t$, $0 \leq t \leq 1$. As in (1), from Lemma 5.7 we get an SSE through positive real matrices from A to B . We denote these

matrices in order as $A = B_0, B_1, \dots, B_\ell = A_1 = B$. For $1 \leq i \leq \ell$, we have from Lemma 5.7 a conjugacy $B_i = E_i^{-1}B_{i-1}E_i$, with E_i close enough to I that $E_i^{-1}B_{i-1}$ and $B_{i-1}E_i$ are positive, which guarantees that there is an ESSE from B_{i-1} to B_i . We also have $E_1E_2 \cdots E_\ell = H_1$.

Now for $1 \leq i \leq \ell$ we will choose a basic elementary matrix E'_i over \mathcal{U} , set $B'_0 = A$, and recursively define $B'_i = E_i^{-1}B_{i-1}E_i$, $1 \leq i \leq \ell$. Define $H'_1 = E'_1E'_2 \cdots E'_\ell$ and $A'_1 = B'_\ell$. Set $V = (H'_1)^{-1}G_1 \in \text{GL}(n, \mathcal{U})$. Then $V^{-1}A'_1V = B$. Let $d = [\det(G_1)]^{1/n}$. Then $H_1 = (1/d)G_1$ and

$$\begin{aligned} V &= \left((H'_1)^{-1} - (H_1)^{-1} \right) G_1 + (H_1)^{-1} G_1 \\ &= \left((H'_1)^{-1} - (H_1)^{-1} \right) G_1 + dI . \end{aligned}$$

Now choose $\epsilon > 0$ for B as in the statement of Lemma 5.8. We choose the E'_i sufficiently close to the E_i to guarantee

- B'_i is positive and ESSE over \mathcal{U}_+ to B'_{i-1} , for $1 \leq i \leq \ell$, and
- $\|V - dI\| < \epsilon$.

We have A'_1 SSE over \mathcal{U}_+ through positive matrices to A . It remains now to show A'_1 is SSE over \mathcal{U}_+ through positive matrices to B . This now follows from Lemma 5.8.

(3) Again find a path G_t in $\text{GL}(n, \mathbb{R})$ such that $A_t = G_t^{-1}AG_t$. By assumption, there is a Y in $\text{GL}(n, \mathcal{U})$ such that $Y^{-1}AY = B$. Therefore $YG_1^{-1} \in \text{Cent}_{\mathbb{R}}(A)$. By assumption there is a matrix Q in $\text{GL}(n, \mathcal{U})$ such that Q and YG_1^{-1} are in the same connected component of $\text{Cent}_{\mathbb{R}}(A)$. Let $W = Q^{-1}Y \in \text{GL}(n, \mathcal{U})$. Then WG_1^{-1} is in the connected component of the identity in $\text{Cent}_{\mathbb{R}}(A)$, and $W^{-1}AW = B$. Therefore (3) follows from (2).

(4) This claim follows from (3) and Lemma 4.2. \square

Remark 5.11. For a given $n \times n$ matrix A , the set of all positive matrices conjugate to A has only finitely many connected components. This is an observation of Sompong Chuysurichay [6, Theorem 1.4.2], made in the language of invariant tetrahedra (discussed in the appendix C). It holds because the set of matrices conjugate to a given matrix can be defined by finitely many inequalities in finitely many variables, and a semialgebraic set has only finitely many connected components [1, Theorem 2.4.4]. Chuysurichay [6, Introduction] pointed out the following corollary of this fact and Theorem 5.10(1) (which was proved in [14]). We record this fact as the following theorem.

Theorem 5.12 ([6, 14]). *Suppose A is a positive $n \times n$ matrix. The collection of positive $n \times n$ matrices conjugate over \mathbb{R} to A contains only finitely many SSE- \mathbb{R}_+ classes.*

Remark 5.13. Note, the set of matrices of a given size which are SSE- \mathbb{R}_+ to a given matrix is not a priori semialgebraic when the lag is unbounded. Indeed, in contrast to Corollary 5.12, Chuysurichay gave an example [6, Theorem 1.9.1] of a connected component \mathcal{C} in a conjugacy class of positive 2×2 real matrices such that the lag of the SSE over \mathbb{R}_+ , guaranteed to exist between any two matrices in \mathcal{C} by Theorem 5.10(1) above (which was proved in [14]), cannot be uniformly bounded in \mathcal{C} . (The unboundedness of the lag arises for a component of positive conjugate matrices whenever there is a matrix on its boundary with more than one irreducible component.)

The fact that the components method produces the finiteness result (5.12) despite the possibility of unbounded lag is an indication of the power of the method.

Remark 5.14. There are examples [2, Appendix E] of primes p in \mathbb{Z} and primitive matrices over $\mathcal{U} = \mathbb{Z}[1/p]$ which are SE over \mathcal{U}_+ (and hence SSE over \mathcal{U} , since \mathcal{U} is a principal ideal domain) but are not SSE over \mathcal{U}_+ . (We do not know whether these examples are SSE over \mathbb{Q}_+ or \mathbb{R}_+ .) There are no positive matrix examples known, for any nondiscrete unital subring \mathcal{U} of \mathbb{R} , of matrices which are SSE over \mathcal{U} but not SSE over \mathcal{U}_+ . The examples [2, Appendix E], based on the work over \mathbb{Z} in [18], are matrices with zero trace, and the general method relies in a fundamental way on the existence of certain matrix powers having zero trace.

Unfortunately, if p is a prime in \mathbb{Z} , then the ring $\mathbb{Z}[1/p]$ does not satisfy the ideal hypothesis of Theorem 5.10(4), and the Centralizer Condition 5.5 is not satisfied by $\mathbb{Z}[1/p]$ (Example 4.3). Therefore Theorem 5.10 does not rule out the possibility that for some p there are positive matrices SSE over $\mathbb{Z}[1/p]$ which are not SSE over $(\mathbb{Z}[1/p])_+$, even in the case the matrices are connected by a path of positive conjugate matrices.

The rest of this section is devoted to generalizing Theorem 5.12 to arbitrary dense subrings of \mathbb{R} . To prepare, we need more definitions. Let \mathcal{U} be a dense subring of \mathbb{R} . Suppose A and B are matrices over \mathcal{U} ; $W \in \mathrm{GL}(n, \mathcal{U})$; and $W^{-1}AW = B$. Given a path $\mathcal{P} = (A_t)_{0 \leq t \leq 1}$ of positive conjugate matrices from A to B , let G be a matrix such that there is a path (G_t) in $\mathrm{GL}(n, \mathbb{R})$ such that $G_t^{-1}AG_t = A_t$, $0 \leq t \leq 1$, with $G_0 = I$ and $G_1 = G$. Let $\pi_0^{\mathcal{U}}(\mathrm{Cent}_{\mathbb{R}}(A))$ denote the subgroup of $\pi_0(\mathrm{Cent}_{\mathbb{R}}(A))$ consisting of those connected components which contain a matrix with all entries in \mathcal{U} . Define $\pi_0(\mathcal{P}, W)$ to be the connected component of $\pi_0(\mathrm{Cent}_{\mathbb{R}}(A))$ containing WG^{-1} . This component is uniquely determined by \mathcal{P} and W . Finally, let $\overline{\pi_0, \mathcal{U}}(\mathcal{P}, W)$ be the coset of $\pi_0^{\mathcal{U}}(\mathrm{Cent}_{\mathbb{R}}(A))$ in $\pi_0(\mathrm{Cent}_{\mathbb{R}}(A))$ which contains WG^{-1} . (We remark as an aside that the coset space $\pi_0(\mathrm{Cent}_{\mathbb{R}}(A))/\pi_0^{\mathcal{U}}(\mathrm{Cent}_{\mathbb{R}}(A))$ is a group, because the group $\pi_0(\mathrm{Cent}_{\mathbb{R}}(A))$ is abelian, because all its elements have order two.)

Lemma 5.15. *Let \mathcal{U} be a dense subring of \mathbb{R} . Suppose A, A_1, A_2 are positive matrices over \mathcal{U} and for $i = 1, 2$ that*

- W_i is a matrix in $\mathrm{GL}(n, \mathcal{U})$ such that $(W_i)^{-1}AW_i = A_i$
- \mathcal{P}_i is a path of positive conjugate matrices from A to A_i .

Suppose $\overline{\pi_0, \mathcal{U}}(\mathcal{P}_1, W_1) = \overline{\pi_0, \mathcal{U}}(\mathcal{P}_2, W_2)$.

Then A_1 and A_2 are SSE over \mathcal{U}_+ .

Proof. For each path \mathcal{P}_i , let G_i be as in the preceding definitions, with $W_i G_i^{-1}$ in the coset $\pi_0(\mathcal{P}_i, W)$. We get a path \mathcal{P} of positive conjugate matrices from A_1 to A_2 by composing the reversal of \mathcal{P}_1 with \mathcal{P}_2 . Let $W = W_1^{-1}W_2$; then $W^{-1}A_1W = A_2$. For the path \mathcal{P} , we have $G = G_1^{-1}G_2$. We compute

$$\begin{aligned} WG^{-1} &= W_1^{-1}W_2G_2^{-1}G_1 = \left(W_1^{-1}(G_1W_1^{-1})W_1\right) \left(W_1^{-1}(W_2G_2^{-1})W_1\right) \\ &= W_1^{-1}CW_1 \end{aligned}$$

where $C = (W_1G_1^{-1})^{-1}(W_2G_2^{-1})$. There is a matrix V over \mathcal{U} which lies in the connected component of $\mathrm{Cent}_{\mathbb{R}}(A)$ containing C , and therefore the connected component of $\mathrm{Cent}_{\mathbb{R}}(A_1)$ containing WG^{-1} contains the matrix $V' = W_1^{-1}CW_1$ from

$\mathrm{GL}(n, \mathcal{U})$. Now $((V')^{-1}W)G$ is in the connected component of the identity in $\mathrm{Cent}_{\mathbb{R}}(A_1)$ and $(V')^{-1}W \in \mathrm{GL}(n, \mathcal{U})$. It follows from the Path Theorem 5.10(2) that A_1 and A_2 are SSE over \mathcal{U}_+ . \square

The number $\gamma(A)$ below was defined in Definition (4.4).

Theorem 5.16. *Let \mathcal{U} be a dense subring of \mathbb{R} . Suppose \mathcal{C} is a path connected set of positive, conjugate $n \times n$ matrices containing a matrix A over \mathcal{U} . Then the number of distinct SSE- \mathcal{U}_+ classes of matrices which contain a matrix in \mathcal{C} which is conjugate to A over \mathcal{U} is finite and cannot exceed*

$$|\pi_0(\mathrm{Cent}_{\mathbb{R}}(A))|/|\pi_0^{\mathcal{U}}(\mathrm{Cent}_{\mathbb{R}}(A))|$$

which is not greater than $2^{\gamma(A)-1}$.

Consequently, the set of positive matrices conjugate over \mathcal{U} to A intersects only finitely many SSE- \mathcal{U}_+ classes.

Proof. The upper bound by the displayed ratio follows from the lemma and the pigeonhole principle. The bound $2^{\gamma(A)-1}$ follows from Proposition 4.5 and the observation that $|\pi_0^{\mathcal{U}}(\mathrm{Cent}_{\mathbb{R}}(A))| \geq 2$ (since $-I \in \mathrm{Cent}_{\mathbb{R}}(A)$). The final claim follows from the lemma and the fact that the set of positive matrices conjugate over \mathbb{R} to A contains only finitely many connected components. \square

6. FINDING POSITIVE PATHS: THE CASE OF ONE NONZERO EIGENVALUE

Definition 6.1. A matrix A is *eventually rank m* if it is square and $\mathrm{rank}(A^k) = m$ for all large k . That A has eventual rank 1 means that its characteristic polynomial has the form $\chi_A(t) = t^m(t - \lambda)$ with λ nonzero.

Lemma 6.2. *Suppose A and B are positive $n \times n$ real matrices with spectral radius λ . Let ℓ_A, r_A be positive left, right eigenvectors of A . Suppose there is $U \in \mathrm{GL}(n, \mathbb{R})$ with positive determinant such that $U^{-1}AU = B$ and the eigenvector $\ell_A U$ of B is positive.*

Then there is a path $\{U_t\}_{0 \leq t \leq 1}$ in $\mathrm{GL}(n, \mathbb{R})$ with $U_0 = I$ and $U_1 = U$ such that for $0 \leq t \leq 1$, the vectors $\ell_A U_t$ and $U_t^{-1}r_A$ are positive eigenvectors for eigenvalue λ for the matrix $A_t = U_t^{-1}AU_t$.

Proof. After passing to $(1/\lambda)A$ and $(1/\lambda)B$, without loss of generality we can suppose $\lambda = 1$. Let D be the diagonal matrix such that $D(i, i) = r_A(i)$. Then $D^{-1}AD$ is stochastic (the right eigenvector has every entry 1). Let $D_t = (1-t)I + tD$. Then $\{D_t^{-1}AD_t\}_{0 \leq t \leq 1}$ is a path of positive matrices from A to a positive stochastic matrix. The same argument holds for B , so without loss of generality we may suppose A and B are stochastic, with positive right eigenvector $r = r_A$ having every entry 1.

Because the subspace of row vectors $W = \{v \in \mathbb{R}^n : vr = 0\}$ is the annihilator of r , the matrix U maps W to W . From the assumptions, if \mathcal{B} is a basis of W , then the matrix representing the restriction to W of U with respect to \mathcal{B} must have positive determinant. Because there is a path from the identity to this matrix in $\mathrm{SL}(n-1, \mathbb{R})$, there is a path $\{T_t\}_{0 \leq t \leq 1}$ of invertible linear transformations $T_t : W \rightarrow W$ such that $T_0 = I$ and $T_1 = U|_W$.

Now we determine the required path of matrices, $\{U_t\}_{0 \leq t \leq 1}$, by specifying the corresponding linear transformations. For $w \in W$, set $wU_t = T_t(w)$. Also require $\ell_A U_t = (1-t)\ell_A + t\ell_A U := \ell_t$. Then $\ell_t > 0$ for all t . Because W contains

no positive vector and W has codimension one, the matrices U_t are well defined and invertible. The vectors $\ell_A U_t$ and $U_t^{-1} r_A := r_t$ are eigenvectors of A_t for the eigenvalue 1. If $w \in W$, then there is a w' in W such that $w' U_t = w$, and therefore $w r_t = w' U_t U_t^{-1} r_A = w' r_A = 0$. Since W has codimension 1, there must be a constant c_t such that $r_t = c_t r$. Because $0 < \ell_A r = \ell_a U_t U_t^{-1} r = \ell_t c r = c \ell_t r$, we conclude $c > 0$. Consequently, both ℓ_t and r_t are positive, as required. Clearly, $U_0 = I$ and $U_1 = U$. \square

For the next lemma, we note that if M is a nilpotent real matrix and $c \neq 0$, then M is conjugate to cM . For a concrete example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^2 \end{pmatrix} = c \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lemma 6.3. *Suppose A and B are positive eventually rank one matrices with nonzero eigenvalue λ , and there is a path $(U_t)_{0 \leq t \leq 1}$ in $\mathrm{GL}(n, \mathbb{R})$ such that $U_0 = I$, $U_1^{-1} A U_1 = B$ and for each $A_t = U_t^{-1} A U_t$, the left and right eigenvectors of A_t are positive.*

Then there is a path $(V_t)_{0 \leq t \leq 1}$ in $\mathrm{GL}(n, \mathbb{R})$ such that $V_0 = I$, $V_1 = U_1$ and each matrix $V_t^{-1} A V_t$ is positive.

Proof. Without loss of generality, suppose $\lambda = 1$. Let ℓ_t and r_t be the left and right positive eigenvectors of A_t , normalized so that $\ell_t r_t = (1)$. Let $P_t = r_t \ell_t$. Let Q_t be the nilpotent matrix such that $A_t = P_t + Q_t$. Then $P_t > 0$, $P_t Q_t = Q_t A_t = 0$, $U_t^{-1} P_0 U_t = P_t$ and $U_t^{-1} Q_0 U_t = Q_t$.

Along the path, the entries of the P_t have a positive lower bound m and the absolute values of entries of the Q_t have a positive upper bound M . Choose a positive $\epsilon < m/M$. Then we have a path of positive conjugate matrices $P_t + \epsilon Q_t$ from $P_0 + \epsilon Q_0$ to $P_1 + \epsilon Q_1$. Taking s from ϵ to 1, we get a path of positive conjugate matrices from $P + \epsilon Q_0$ to $P + Q_0 = A$, and likewise from $P_1 + \epsilon Q_1$ to $P_1 + Q_1 = B$. Composing paths, we get a path of positive conjugate matrices from A and B . Reparametrizing, we get the path $(V_t)_{0 \leq t \leq 1}$ such that $V_0 = I$ and $V_1 = U_1$. \square

The next result was proved in [14] for the case $\mathcal{U} = \mathbb{Q}$ or \mathbb{R} . The positive matrix path construction below is a matrix version of the invariant tetrahedra argument in [14]. We describe the approach from [14] of “positive invariant tetrahedra” in Appendix C.

Theorem 6.4. *Suppose \mathcal{U} is a nondiscrete unital subring of \mathbb{R} , and A and B are nonnegative eventually rank one matrices which are SSE over \mathcal{U} .*

Then A and B are SSE over \mathcal{U}_+ .

Proof. After passing to matrices SSE over \mathcal{U}_+ , we may assume that A and B are primitive (using the eventually rank one assumption), and then positive (by Proposition B.3). By Theorem 3.1, we may assume also we have $U \in \mathrm{GL}(n, \mathcal{U})$ such that $AU = UB$, $\det U > 0$ and U sends a positive eigenvector of A to a positive eigenvector of B . By Lemmas 6.2 and 6.3, there is a path $(U_t)_{0 \leq t \leq 1}$ in $\mathrm{GL}(n, \mathbb{R})$ such that $U_0 = I$, $U_1 = U$ and each $U_t^{-1} A U_t$ is positive. By Theorem 5.10(2), it follows that A and B are SSE over \mathcal{U}_+ . \square

Remark 6.5. The one eigenvalue result above looks better in contrast to the lack of other general results. For every subring \mathcal{U} of \mathbb{R} , for every primitive matrix A over

\mathcal{U} , it is unknown whether there exists an algorithm which given B primitive and SSE over \mathcal{U} to A decides whether B is SSE over \mathcal{U}_+ to A .

Theorem 6.4 is not a complete solution to the problem of classifying eventually rank one positive matrices over \mathcal{U} (for an arbitrary dense subring of \mathbb{R}). It is complete with regard to addressing positivity, but we do not understand in general how SSE refines SE over \mathcal{U} . Especially,

Problem 6.6. *Suppose \mathcal{U} is a nondiscrete unital subring of \mathbb{R} , and A, B are eventually rank one matrices which are shift equivalent over the ring \mathcal{U} . Must they be strong shift equivalent over \mathcal{U} ?*

However, we are able to handle some classes of rings, as follows.

Theorem 6.7. *Suppose \mathcal{U} is a nondiscrete unital subring of \mathbb{R} , and A is a non-negative eventually rank one matrix over \mathcal{U} , with nonzero eigenvalue λ . Then the following hold.*

- (1) *If \mathcal{U} is a Dedekind domain and A is shift equivalent over \mathcal{U} to the matrix (λ) , then A is SSE over \mathcal{U}_+ to (λ) .*
- (2) *If \mathcal{U} is a principal ideal domain (e.g., a field), then A is SSE over \mathcal{U}_+ to (λ) .*

Proof. (1) Over a Dedekind domain \mathcal{U} , SE- \mathcal{U} implies SSE- \mathcal{U} [3, Prop. 2.4], so Theorem 6.4 applies.

(2) Over the principal ideal domain \mathcal{U} , A is SSE- \mathcal{U} to a nonsingular matrix [7, 31]. This matrix can only be (λ) , so again Theorem 6.4 applies. \square

Remark 6.8. Example 2.2 in [3] provides a 2×2 positive matrix A' with eigenvalues 0 and 1 over the Dedekind domain $\mathcal{U} = \mathbb{Z}[\sqrt{15}]$ which is not shift equivalent over \mathcal{U} to a nonsingular matrix.

Whenever a Dedekind domain \mathcal{U} is not a principal ideal domain, there will be matrices over \mathcal{U} which are not SSE- \mathcal{U} to a nonsingular matrix [3].

Remark 6.9. The proofs above easily adapt to prove the result stated next, which is one version of the “positive models” result in [16].

Theorem 6.10. *Suppose A and B are $n \times n$ positive real matrices, and there are matrices P, Q_0, Q_1, U such that the following hold.*

- P is a positive matrix
- A and B are internal direct sums, $A = P + Q_0$ and $B = P + Q_1$, with the matrices Q_0, Q_1 nilpotent
- There is $U \in \text{SL}(n, \mathbb{R})$ such that $UP = PU$ and $UQ_0 = Q_1U$.

Then there is a path of positive matrices $A_t = U_t^{-1}AU_t$ from $A = A_0$ to $B = A_1$, such that $U_0 = I$ and $U_1 = U$.

Theorem 6.10 looks like a powerful tool, but so far it has not led to a general result.

Problem 6.11. *Suppose A is a positive real matrix. Must there exist a positive matrix P , with $\text{rank}(P) = \text{rank}(P^2)$, and a nilpotent matrix Q , such that $PQ = QP = 0$ and A is strong shift equivalent over \mathbb{R}_+ to $P + Q$?*

7. THE CONNECTION THEOREM

Below, $\|C\|_{\max}$ denotes the maximum absolute value of an entry of C .

Definition 7.1. For an $n \times n$ real matrix A , and $\epsilon > 0$, $\mathcal{N}_\epsilon(A)$ denotes the set of $n \times n$ matrices B such that $\|B - A\|_{\max} < \epsilon$.

Definition 7.2. For an $n \times n$ real matrix A and $\epsilon > 0$, $\mathcal{N}_\epsilon^{\text{SE}}(A)$ denotes the set of $n \times n$ matrices B which are shift equivalent over \mathbb{R} to A and satisfy $\|B - A\|_{\max} < \epsilon$.

Theorem 7.3 (Connection Theorem). *Suppose A is an $n \times n$ positive matrix. Then there is a $\delta > 0$ such that for any B, C in $\mathcal{N}_\delta^{\text{SE}}(A)$ and $m \geq n^2/2$, there are row splittings of B, C to positive conjugate matrices B', C' such that there exists a path of positive conjugate matrices from B' to C' , and therefore the matrices B, C are SSE- \mathbb{R}_+ , through positive matrices not larger than $(n^2/2) \times (n^2/2)$.*

Moreover, if B and C have their entries in a nondiscrete subring \mathcal{U} of \mathbb{R} , then the splittings to B' and C' can be done through matrices over \mathcal{U}_+ . If in addition \mathcal{U} is a field, then the matrices B, C are SSE- \mathcal{U}_+ , through positive matrices not larger than $(n^2/2) \times (n^2/2)$.

In the Connection Theorem, $A = B$ is allowed. Before proving the theorem, we record some immediate consequences.

Corollary 7.4. *If A is a positive $n \times n$ matrix and $\dim(\ker(A)) \geq 1$, then A is SSE over \mathbb{R}_+ to a positive $n \times n$ matrix B such that $\dim(\ker(B)) = 1$.*

Proof. Given $\dim(\ker(A)) > 1$, there are positive matrices shift equivalent to A which are arbitrarily close to A such that $\dim(\ker(A)) = 1$, as one can see by replacing each superdiagonal zero in the Jordan form of the nilpotent part of A with ϵ . Therefore Theorem 7.5 applies to prove the corollary. \square

The Path Theorem 5.10 produced SSE's over \mathbb{R}_+ from paths of positive matrices which are conjugate. The following consequence of the Connection Theorem shows we only need those matrices to be shift equivalent.

Theorem 7.5. *Suppose $(A_t), 0 \leq t \leq 1$, is a path of positive shift equivalent $n \times n$ matrices. Then A_0 and A_1 are SSE over \mathbb{R}_+ .*

Proof of Theorem 7.5. It follows from compactness that for $0 \leq t \leq 1$, the Connection Theorem holds for A_t in place of A , for a uniform ϵ (independent of t). Consequently A_0 and A_1 are SSE over \mathbb{R}_+ . \square

The rest of this section is devoted to the proof of the Connection Theorem, which relies also on Theorem D.2. The consequences of the Connection Theorem in later sections can be read independent of the proof of Theorem D.2.

We prepare for the proof with two lemmas. The idea behind Lemma 7.6, apart from the generality of \mathcal{U} , can be found in [17] and [14, Lemma 1]. J_k denotes the standard $k \times k$ Jordan block matrix (zero except for entries 1 in positions $(i, i + 1)$, $1 \leq i < k$). $J_0(A)$ denotes the nilpotent part of the Jordan form of a matrix A .

Lemma 7.6. *Suppose \mathcal{U} is a nondiscrete subring of \mathbb{R} , $n \in \mathbb{N}$, A and B are positive $n \times n$ matrices over \mathcal{U} , $\|A - B\| < \epsilon$, $0 < \alpha < 1$ and the i th rows of A and B are denoted w_A and w_B . Let A' be the $(n + 1) \times (n + 1)$ matrix obtained by a splitting of its i th row corresponding to $w = \alpha w + (1 - \alpha)w$. Suppose B is an $n \times n$ positive matrix. Then there is an $(n + 1) \times (n + 1)$ positive matrix B' , obtained by a splitting of its i th row, such that the following hold.*

- (1) $\|B' - A'\| < \epsilon$
- (2) $J_0(B')$ can be chosen to be either (i) $J_0(B) \oplus [0]$ or, (ii) for any k such that $J_0(A)$ has a $k \times k$ Jordan block, $J_0(B')$ can be obtained from $J_0(B)$ by replacing a $k \times k$ Jordan block with a $(k+1) \times (k+1)$ Jordan block.
- (3) If B has entries in \mathcal{U} , then the splitting to B' can be done over \mathcal{U} .

Proof. Without loss of generality, we may assume $\alpha \in \mathcal{U}$. For (i), use the splitting of row i by $\alpha w + (1 - \alpha)w$. For (ii), we have by assumption that there is a vector v such that $vB^k = 0, vB^{k-1} \neq 0$ and v is not in the image of A . Let \mathbb{F} denote the field of fractions of \mathcal{U} . The kernel of B^k contains a dense subset of vectors from \mathbb{F}^n . Pick v' in $\ker(B^k) \cap \mathcal{U}^n$ with $\|u - v'\|$ small enough that $v'B^{k-1} \neq 0$ and v' is not in the image of B . Pick $\beta \neq 0$ from \mathcal{U} such that $\beta v' \in \mathcal{U}^n$. Pick $\gamma > 0$ in \mathcal{U} arbitrarily small, and small enough that for $s := \alpha w_B + \gamma \beta v$ we have $s > 0$ and $w_B - s > 0$. Now form B' by splitting row i of B according to $w_B = s + (w_B - s)$. Then B' is conjugate to $\begin{pmatrix} B & 0 \\ s & 0 \end{pmatrix}$ and to $\begin{pmatrix} B & 0 \\ \gamma \beta v' & 0 \end{pmatrix}$. This last matrix has the required Jordan form. For small γ , we have $\|B' - A'\| < \epsilon$. \square

For the next lemma we establish some notation. For a square matrix M , we let G_M, H_M denote the unique matrices G, H such that $M = G + H, GH = HG = 0, H$ is nilpotent and $\text{rank}(G) = \text{rank}(G^2)$. We also use U_M, N_M, F_M to denote matrices such that $U_M^{-1} M U_M = F_M \oplus N_M$, where F_M is nonsingular and N_M is nilpotent, and for concreteness N_M is in Jordan form. In this case, $U_M^{-1} G_M U_M = F_M \oplus 0$ and $U_M^{-1} H_M U_M = 0 \oplus N_M$, where 0 denotes a zero matrix of appropriate size. The norm used below is the max norm.

Finally, we define a notion critical for our proof of the Connection Theorem.

Definition 7.7. Suppose N is an $n \times n$ nilpotent matrix and \mathcal{C} is a conjugacy class of $n \times n$ nilpotent matrices. We say \mathcal{C} is *locally connected at N* if for every $\epsilon > 0$ there exists $\delta > 0$ such that any two matrices in $\mathcal{C} \cap \mathcal{N}_\delta(N)$ are connected by a path in $\mathcal{C} \cap \mathcal{N}_\epsilon(N)$.

Lemma 7.8. Suppose $\epsilon > 0$ and A, B are positive $n \times n$ matrices such that G_A and G_B are conjugate, and the conjugacy class \mathcal{C} of N_B is locally connected at N_A .

Then there is a $\delta > 0$ such that any two matrices in $\mathcal{N}_\delta(A)$ which are conjugate to B are connected by a path of positive conjugate matrices in $\mathcal{N}_\epsilon(A)$.

Proof. Fix a matrices U, F_A, N_A such that $U^{-1} A U = F_A \oplus N_A$. The idea of the proof is the following. For δ small enough, given two matrices conjugate to B inside $\mathcal{N}_\delta(A)$, we show there are paths from them in $\mathcal{N}_\epsilon(A)$ to matrices C_1, C_2 which are conjugated by U to matrices $C'_1 = \begin{pmatrix} F_A & 0 \\ 0 & N_1 \end{pmatrix}$ and $C'_2 = \begin{pmatrix} F_A & 0 \\ 0 & N_2 \end{pmatrix}$ such that there is a path of conjugate matrices from N_1 to N_2 which induces a path from C'_1 to C'_2 which U^{-1} conjugates to the desired path from C_1 to C_2 . We spell out quantifiers for this next, for a matrix C conjugate to B .

Take ϵ to be smaller than $\|A\|$. Pick $\epsilon_1 > 0$ such that $\|N' - N_A\| < \epsilon_1$ implies $\|U(F_A \oplus N')U^{-1} - A\| < \epsilon$. Pick $\epsilon_2 > 0$ such that if conjugate nilpotent matrices N_1, N_2 are in $\mathcal{N}_{\epsilon_2}(N_A)$, then there is a path of conjugate matrices in $\mathcal{N}_{\epsilon_2}(N_A)$ from N_1 to N_2 . Pick $\epsilon_3 > 0$ such that $\|X - A\| < \epsilon_3$ implies $\|U^{-1}(X - A)U\| < \epsilon_2$.

Finally, pick $\delta > 0$ such that if C is conjugate to B and $\|A - C\| < \delta$, then the conjugate matrices A^n and C^n are sufficiently close that there is a V in $\text{SL}(n, \mathbb{R})$ such that $V^{-1} A^n V = C^n$ (which means $V^{-1}(G_A)^n V = (G_C)^n$) and $\|V - I\|$ is sufficiently small that the following hold:

- (1) $V^{-1}G_AV = G_C$.
- (2) $\|V^{-1}CV - A\| < \epsilon_3$.
- (3) There is a path (V_t) in $\text{SL}(n, \mathbb{R})$ from $I = V_0$ to $V = V_1$ remaining sufficiently close to I that $V_t^{-1}CV_t \in \mathcal{N}_\epsilon(A)$, $0 \leq t \leq 1$.

Because V maps $\ker(C^n)$ onto $\ker(A^n)$, the matrix $U^{-1}(V^{-1}BV)U$ has the form $\begin{pmatrix} F_A & 0 \\ 0 & N' \end{pmatrix}$, with

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & N' \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & N_A \end{pmatrix} &= \begin{pmatrix} F_A & 0 \\ 0 & N' \end{pmatrix} - \begin{pmatrix} F_A & 0 \\ 0 & N' \end{pmatrix} \\ &= U^{-1}(V^{-1}BV - A)U . \end{aligned}$$

Since $\|V^{-1}BV - A\| < \epsilon_3$, we have

$$\|N' - N_A\| = \left\| \begin{pmatrix} 0 & 0 \\ 0 & N' \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & N_A \end{pmatrix} \right\| < \epsilon_2$$

which shows our δ is small enough to establish the conclusion of the lemma. \square

Proof of the Connection Theorem. If B and A are conjugate, then the theorem follows from the Path Theorem. So we assume B and A are not conjugate, which implies $n \geq 3$.

The strategy of the proof is to take a row splitting of A to a suitable positive matrix A' chosen independent of B and C ; pick a suitable class \mathcal{C} of nilpotent matrices which is locally connected at $N_{A'}$; and then for $\delta > 0$ taken from Lemma 7.8, perform row splittings of B and C to matrices B' and C' in $\mathcal{C} \cap \mathcal{N}_\delta(A')$.

To begin the proof, we choose A' and a sequence of matrices A_i , $n \leq i \leq m$, such that $A = A_n$, $A' = A_m$ and for $n \leq i < m$, the matrix A_{i+1} is obtained by splitting a row of A_i into two proportional rows. For example, if $m = sn + p$ with $s \geq 1$ and $1 \leq p < n$, then we could split each of the first p rows into s equal rows and split each of the remaining rows into $s - 1$ equal rows. Then $\text{rank}(A') = \text{rank}(A)$, and A' is conjugate to $A \oplus 0_{m-n}$.

Using notations from Definition D.1, we define $h = \max\{h(A), h(B), h(C)\}$ and $\beta = \max\{\beta(A), \beta(B), \beta(C)\}$. (If B is in $\mathcal{N}_\epsilon^{\text{SE}}(A)$ and ϵ is sufficiently small, then $h(B) \geq h(A)$ must hold.) Let r be the rank of F_A . For any C shift equivalent to A over \mathbb{R} , we have F_C conjugate to F_A and $\text{rank}(F_C) = r$. So, if C is $m \times m$ and shift equivalent to A , then C is conjugate to $F_A \oplus N_C$, and the nilpotent matrix N_C is $(m - r) \times (m - r)$. We will define \mathcal{C} to be the conjugacy class of $(m - r) \times (m - r)$ matrices which contains the matrix $(J_h)^\beta \oplus 0_q$, where q is $m - r - \beta h$. We need to check this makes sense ($q \geq 0$) and also that $q > 0$. Because $n \geq 3$ and

$$(\beta)(h) \leq \left(\frac{n-r}{2}\right)(n-r-1) = \frac{n^2}{2} - \frac{r}{2}(3n-1) ,$$

we have

$$\begin{aligned} (\beta)(h) + r &\leq \frac{n^2}{2} - \frac{r}{2}(3n-3) \\ &\leq \frac{n^2}{2} - \frac{1}{2}(3(3)-1) = \frac{n^2}{2} - 3r < \frac{n^2}{2} \leq m , \end{aligned}$$

and therefore $q > 0$. Clearly

- (1) \mathcal{C} has a zero Jordan block (because $q > 0$).
- (2) $h(\mathcal{C}) = h \geq h(A) = h(A')$.

$$(3) \beta^{\text{top}}(\mathcal{C}) = \beta \geq \beta(A) = \beta(A') .$$

It follows from Theorem D.2 that \mathcal{C} is locally connected at $N_{A'}$. Therefore, we can specify any $\epsilon > 0$ and for that ϵ pick $\delta > 0$ in the statement of Lemma 7.8. This will be the δ in the statement of the Connection Theorem.

Now we describe the splitting of B to B' (the argument to split C to C' is the same), such that $N_{B'} \in \mathcal{C}$. Starting with $B_n = B$ and $\|B - A\| < \delta$, for $n \leq i < m$ we inductively appeal to Lemma 7.6 to split B_i to B_{i+1} , with $\|A_{i+1} - B_{i+1}\| < \delta$. We use condition (2) of Lemma 7.6 at each stage as follows, applying the first listed criterion for which B_i satisfies the required condition.

- (1) If N_{B_i} has no zero Jordan block, then $N_{B_{i+1}} = N_{B_i} \oplus 0_1$.
- (2) If N_{B_i} has fewer than β Jordan blocks which are 2×2 or larger, then $N_{B_{i+1}}$ is N_{B_i} with a zero block replaced by 2×2 Jordan block.
- (3) If N_{B_i} has a $k \times k$ Jordan block of with $2 \leq k < h$, then for a maximum such k , $N_{B_{i+1}}$ is N_{B_i} with a Jordan k -block replaced by a $(k+1)$ -block.
- (4) If N_{B_i} has β Jordan blocks which are $h \times h$, then $N_{B_{i+1}} = N_{B_i} \oplus 0_1$.

Clearly B' has the required form, and there is a path of positive conjugate matrices from B' to C' . By the Path Theorem (5.10), B' and C' are SSE over \mathbb{R}_+ , through positive $m \times m$ matrices.

The ‘‘Moreover’’ condition of keeping splittings over \mathcal{U} can be achieved by condition (3) of Lemma 7.6. If \mathcal{U} is a field, then the conjugacy of B' and C' over \mathbb{R} implies their conjugacy over \mathcal{U} , and (again using that \mathcal{U} is a field) by the Path Theorem we have an SSE- \mathcal{U}_+ through positive matrices from B' to C' , and hence also from B to C . This finishes the proof. \square

8. FROM SSE OVER \mathbb{R}_+ TO SSE OVER \mathbb{U}_+

Theorem 8.1. *Suppose A, B are positive matrices over \mathbb{R} which are SSE- \mathbb{R}_+ . Then A, B are SSE over \mathbb{R}_+ through positive matrices: there are positive matrices $A = A_0, A_1, \dots, A_\ell = B$, with A_i elementary SSE over \mathbb{R}_+ to A_{i+1} , $0 \leq i < \ell$.*

Proof. Appealing to Theorem 3.8, choose a primitive matrix C over \mathbb{R}_+ and a nonsingular diagonal matrix D over \mathbb{R}_+ such that C is reached from A by finitely many row splittings through primitive matrices and $D^{-1}CD$ is reached from B by finitely many column splittings through primitive matrices. Then choose a construction, by the procedure described in Appendix B, of a positive matrix \tilde{C} SSE over \mathbb{R}_+ to C . Appealing to the Connection Theorem, choose $\delta > 0$ such that matrices shift equivalent to \tilde{C} and δ close to \tilde{C} are SSE to each other through positive matrices. Appealing to Lemma B.4, pick $\nu > 0$ such that for any positive matrix M with $\|M - C\| < \nu$ there is a strong shift equivalence over \mathbb{R}_+ through positive matrices from M to a positive matrix \tilde{M} such that $\|\tilde{M} - \tilde{C}\| < \delta$.

Now, perform row splittings from A through positive matrices to a positive matrix A^* such that $\|A^* - C\| < \nu$. This is done simply by approximating the string of splittings from A to C over \mathbb{R}_+ by a string of splittings from A to C through positive matrices. By composition, we have an SSE over \mathbb{R}_+ from A to a matrix \tilde{A} within δ of \tilde{C} .

The argument to obtain an SSE over \mathbb{R}_+ through positive matrices from B to a matrix \tilde{B} within δ of \tilde{C} is similar. We obtain a positive matrix B^* near $D^{-1}CD$ by approximating the given column splittings from B to $D^{-1}CD$. There is an elementary SSE over \mathbb{R}_+ from B^* to the positive matrix $D^{-1}B^*D := B^{**}$. We take

B^* close enough to $D^{-1}CD$ to guarantee $\|B^{**} - C\| < \nu$. Then we apply Lemma B.4 again to obtain the SSE through positive matrices from B^{**} to the desired \tilde{B} near C . By the Connection Theorem, \tilde{A} and \tilde{B} are SSE over \mathbb{R}_+ through positive matrices. By composing the assembled SSEs, the theorem is proved. \square

Theorem 8.2 below was proved in [17] for $\mathcal{U} = \mathbb{Q}$ under the additional assumption that A and B are SSE over \mathbb{R}_+ through positive matrices.

Theorem 8.2. *Let \mathcal{U} be a subfield of \mathbb{R} . Suppose A, B are positive matrices over \mathcal{U} which are SSE over \mathbb{R}_+ . Then A and B are SSE over \mathcal{U}_+ , through positive matrices.*

Proof. We examine the proof of Theorem 8.1 and check that the SSEs constructed in the various steps can be taken through positive matrices over \mathcal{U} .

The splittings to A^* and B^* can be done over \mathcal{U} .

Approximate D by a diagonal matrix D' over \mathcal{U}_+ . In place of B^{**} , use $(D')^{-1}B^*D' := B'$. Because \mathcal{U} is a field, the positive matrix B' has its entries in \mathcal{U} and is ESSE over \mathcal{U}_+ to B^* (by the matrices $(D')^{-1}$ and B^*D').

The matrices \tilde{A} and \tilde{B} are constructed from A^* and B' (matrices over \mathcal{U}) by appeal to Proposition B.3, and therefore they can be taken over \mathcal{U} . Lemma B.4 allows the approximating SSE through positive matrices to be taken over \mathcal{U} . Because \mathcal{U} is a field, the Connection Theorem then gives an SSE through positive matrices over \mathcal{U} from \tilde{A} to \tilde{B} .

This completes the proof. \square

Theorem 8.3. *Suppose A is a positive $n \times n$ matrix. The collection of positive $n \times n$ matrices conjugate over \mathbb{R} to A contains only finitely many SSE- \mathbb{R}_+ classes.*

Proof. This follows immediately from Theorem 5.12 and Theorem 8.2. \square

Problem 8.4. *Let \mathcal{U} be a dense subring of \mathbb{R} . Suppose A and B are positive matrices which are strong shift equivalent over \mathcal{U} and also over \mathbb{R}_+ . Must they be strong shift equivalent over \mathcal{U}_+ ?*

APPENDIX A. MAKING SSE NONDEGENERATE

The purpose of the appendix is to prove Proposition 2.9, which we now restate.

Proposition 2.9 *Suppose \mathcal{U} is a ring which is torsion free as an additive group. Suppose nondegenerate matrices A and B are SSE over \mathcal{U} . Then they are SSE through a chain of ESSEs $A_{i-1} = R_i S_i, A_{i+1} = S_i R_i$ such that all the matrices A_i, R_i, S_i are nondegenerate.*

Recall, a matrix is nondegenerate if it has no zero row and no zero column. Below, by a nonzero matrix we mean a matrix which is not the zero matrix.

We will prove Proposition 2.9 after proving three lemmas.

Lemma A.1. *Suppose \mathcal{U} is a ring and there are matrices A, B, C, R, S, R', S' over \mathcal{U} satisfying the following conditions $A = RS$, $B = SR$; $B = R'S'$, $C = S'R'$; $A \neq 0$, $B = 0$, $C \neq 0$. Then there are nonzero matrices $A = A_0, A_1, \dots, A_6 = C$ such that A_{i-1} is ESSE over \mathcal{U} to A_i , $1 \leq i \leq 6$.*

Proof. In block form, define

$$\begin{aligned} A_1 &= \begin{pmatrix} RS & 0 \\ S & 0 \end{pmatrix} & A_4 &= \begin{pmatrix} 0 & 0 \\ S' & 0 \end{pmatrix} = \begin{pmatrix} R'S' & 0 \\ S' & 0 \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & 0 \\ S & SR \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} & A_5 &= \begin{pmatrix} 0 & 0 \\ S' & S'R' \end{pmatrix} \\ A_3 &= \begin{pmatrix} 0 & 0 & 0 \\ S & 0 & 0 \\ 0 & S' & 0 \end{pmatrix} & A_6 &= (S'R') = B. \end{aligned}$$

The matrices S and S' cannot be zero (because A and C are not zero), so the A_i are not zero. An ESSE from $A \rightarrow A_1$ is given by

$$A = (I \ 0) \begin{pmatrix} RS \\ S \end{pmatrix}, \quad A_1 = \begin{pmatrix} RS \\ S \end{pmatrix} (I \ 0).$$

There are ESSEs $A_2 \rightarrow A_3, A_3 \rightarrow A_4, A_5 \rightarrow A_6$ of this type or a transpose type. An ESSE $A_1 \rightarrow A_2$ is given by

$$\begin{aligned} A_1 &= \begin{pmatrix} RS & 0 \\ S & 0 \end{pmatrix} = \begin{pmatrix} RS & RSR \\ S & SR \end{pmatrix} \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & 0 \\ S & SR \end{pmatrix} = \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} \begin{pmatrix} RS & RSR \\ S & SR \end{pmatrix} \end{aligned}$$

The remaining ESSE $A_4 \rightarrow A_5$ is of the same type. \square

Lemma A.2. *Suppose \mathcal{U} is a unital semiring and A is ESSE over \mathcal{U} to 0_m , the $m \times m$ zero matrix. Then A is ESSE over \mathcal{U} to 0_{m+k} , for all k in \mathbb{N} .*

Proof. We are given $A = RS, 0_M = SR$. Then $A = (R \ 0) \begin{pmatrix} S \\ 0 \end{pmatrix}$ and $0_{m+k} = \begin{pmatrix} S \\ 0 \end{pmatrix} (R \ 0)$ where 0 denotes a zero block of the necessary size. \square

Lemma A.3. *Let \mathcal{U} be a unital ring which is torsion free as an additive group. Suppose A is an $n \times n$ matrix over \mathcal{U} which is not the zero matrix. Then there is V in $\text{SL}(n, \mathbb{Z})$ such that $V^{-1}AV$ is nondegenerate.*

Proof. We can assume $n > 1$. For example, suppose row 1 of A is nonzero and row n of A is zero. Given $M \in \mathbb{N}$, let E be the basic elementary matrix such that $E(n, 1) = M$ and set $C = EAE^{-1}$. Then

$$\begin{aligned} C(i, 1) &= A(i, 1) - MA(i, n) \quad \text{if } i < n \\ C(n, 1) &= MA(1, 1) - M^2A(1, n) \\ C(n, j) &= MA(1, j) \quad \text{if } j > 1 \\ C(i, j) &= A(i, j) \quad \text{if } i < n \text{ and } j > 1. \end{aligned}$$

Appealing to the torsion free assumption, choose M such that

$$1 \leq i < n \text{ and } A(i, 1) \neq 0 \implies A(i, 1) \neq MA(i, n).$$

Then $A(i, j) \neq 0$ implies $C(i, j) \neq 0$. In addition, row n of C is not zero, as follows. If there exists $j > 1$ with $A(1, j) \neq 0$, then $C(n, j) \neq 0$; otherwise, $A(1, 1)$ is the only nonzero entry of row 1 of A , and $C(n, 1) = MA(1, 1) \neq 0$.

Iterating this move as needed, with other indices (i, j) in place of $(1, n)$, and interchanging the role of column and row as needed, we produce $V \in \text{SL}(n, \mathbb{Z})$ such that $V^{-1}AV$ is nondegenerate.

□

Remark A.4. We are not concerned in this paper with finding the sharpest version of Proposition 2.9. However, we note that Lemma A.3 would be false if the “torsion free” assumption were simply dropped. Over the field $\mathbb{Z}/2$, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $A \neq 0$ but A is not conjugate over $\mathbb{Z}/2$ to a nondegenerate matrix, because B is the only rank one nondegenerate 2×2 matrix over $\mathbb{Z}/2$, and $A^2 \neq 0 = B^2$.

We are now ready to prove Proposition 2.9.

Proof of Proposition 2.9. We are given some string of ESSEs over \mathcal{U} from $A = A_0$ to $B = A_\ell$,

$$A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{\ell-1} \rightarrow A_\ell = B$$

with matrices R_i, S_i over \mathcal{U} such that $A_{i-1} = R_i S_i$, $A_i = S_i R_i$, for $1 \leq i \leq \ell$.

Suppose for some i and some $k > 2$ that A_i and A_{i+k} are not zero, but A_j is a zero matrix for $i < j < i + k$. By Lemma A.2, there is a zero matrix Z ESSE to A_i and to A_{i+k} . We replace the ESSEs $A_i \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A_{i+k}$ with ESSEs $A_i \rightarrow Z \rightarrow A_{i+k}$. After iterating this move as necessary, we may assume that $A_i = 0$ implies $A_{i-1} \neq 0$ and $A_{i+1} \neq 0$.

Then, by Lemma A.1, if $A_i = 0$, we may replace the ESSEs $A_{i-1} \rightarrow A_i \rightarrow A_{i+1}$ with a string of ESSEs from A_{i-1} to A_{i+1} through nonzero matrices. After iterating as needed, we may assume every A_i is not zero.

If $0 < i < \ell$ and $U^{-1}A_iU = A'_i$, then we can replace

$$(A.1) \quad A_{i-1} \xrightarrow{(R_i, S_i)} A_i \xrightarrow{(R_{i+1}, S_{i+1})} A_{i+1}$$

with

$$(A.2) \quad A_{i-1} \xrightarrow{(R_i U, U^{-1} S_i)} A'_i \xrightarrow{(U^{-1} R_{i+1}, S_{i+1} U^{-1})} A_{i+1} \ .$$

Thus by repeated application of this move, with $U^{-1}A_iU = A'_i$ nondegenerate by Lemma A.3, we can pass to an SSE through nondegenerate matrices as required. □

APPENDIX B. BOOLEAN MATRICES AND POSITIVITY

Let \mathcal{U} be a nondiscrete unital subring of \mathbb{R} .

We will include in this section a proof of the result of [12] that every primitive matrix over \mathcal{U} is SSE over \mathcal{U}_+ to a positive matrix. As in [12], this is done by proving the result for Boolean matrices and then carrying it over. We can then prove the approximation result Lemma B.4, which we need in Section 8.

Boolean matrices are matrices with entries in the Boolean semiring $\mathcal{B} = \{0, 1\}$, in which $1 + 1 = 1$. The usual row and column splitting and amalgamations can be used to produce SSEs over \mathcal{B} . In particular, if a row i of A is less than or equal to row j of A , then adding column j of A to column i produces a matrix B SSE over \mathcal{B} to A ; for the corresponding elementary matrix E , we have $A = EA$ and $B = AE$.

An example, assuming row 1 of A is less than or equal to row 2, is

$$\begin{aligned} A &= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = EA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ B &= \begin{pmatrix} a_1 + a_2 & a_2 & a_3 \\ b_1 + b_2 & b_2 & b_3 \\ c_1 + c_2 & c_2 & c_3 \end{pmatrix} = AE = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

If A is the Boolean image of a matrix A' over \mathcal{U}_+ , then there are E', B' over \mathcal{U}_+ with Boolean images E, B such that $A' = E'A'$ and $B' = A'E'$. Here, E' is an elementary matrix whose off diagonal entry can be chosen arbitrarily close to zero, and B' is conjugate over \mathcal{U} to A' . In the example (using the letter entries in A above to denote entries of A' , for simplicity), we have

$$\begin{aligned} B' &= (E')^{-1}A'E' = \begin{pmatrix} 1 & 0 & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 - \epsilon a_1 & b_2 - \epsilon a_2 & b_3 - \epsilon a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + \epsilon a_2 & a_2 & a_3 \\ b_1 - \epsilon a_1 + \epsilon b_2 - \epsilon^2 a_2 & b_2 - \epsilon a_2 & b_3 - \epsilon a_3 \\ c_1 + \epsilon c_2 & c_2 & c_3 \end{pmatrix}. \end{aligned}$$

For any sufficiently small and positive ϵ from \mathcal{U} , we have $(E')^{-1}A' \geq 0$, and therefore an ESSE over \mathcal{U}_+ between A' and $(E')^{-1}A'E'$.

The next result is proved in [12] and we take that proof.

Proposition B.1. *Suppose A is a primitive Boolean matrix with positive trace. Then A is SSE over \mathcal{B} to [1].*

Proof. A is the adjacency matrix of a directed graph. Take a closed walk through the graph which passes through every vertex at least once. Suppose the walk passes through vertex i n_i times. Define a matrix V which has, for each i , n_i copies of row i of A . (Over \mathcal{B} , a row copying is an example of a row splitting.) Let U be the subdivision matrix such that $UV = A$, and set $VA = B$, SSE over \mathcal{B} to A . Then there is a closed walk through the graph of B which hits every vertex exactly once. Without loss of generality, then, suppose B is $m \times m$ and $B(1,1) = 1$ and $P \leq B$, where P is the matrix with positive entries at $(1,1), (1,2), (2,3), \dots, (m,1)$.

Next, define an ESSE from B to a $2m \times 2m$ matrix C , by

$$\begin{aligned} C &= \begin{pmatrix} P & B \\ P & B \end{pmatrix} = \begin{pmatrix} I_m \\ I_m \end{pmatrix} (P \ B) \\ & \quad B = (P \ B) \begin{pmatrix} I_m \\ I_m \end{pmatrix} \end{aligned}$$

An example with C 10×10 is

$$C = \begin{pmatrix} P & B \\ P & B \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & 0 & 0 & \bullet & \bullet & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & 0 & 0 & \bullet & \bullet & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

in which a bullet denotes an entry which could be 0 or 1, depending on A .

Note, column 1 of C is greater than or equal to column 2. So, we may add row 1 of C to row 2 (to produce an SSE matrix). Now in the order $i = 2, 3, \dots, m - 1$, add row i to row $i + 1$. At the point row i is added, column i will be greater than or equal to column $i + 1$, so the addition will give an ESSE. After these moves, row m has every entry 1. In order, for $i = m, m - 1, \dots, 2, 1$, add column i to every other column. At the point column i is added, row i will be all 1's, so SSE is respected. Because every row has an entry 1 in one of the columns $1, 2, 3, \dots, m$, at the conclusion of this C will be transformed to a matrix with every entry 1. Such a matrix equals vv^{tr} , where v is a column vector with every entry 1, and then $v^{tr}v = [1]$. \square

The next result extends a result in [15], with essentially the same proof. Let $\omega(n)$ denote the maximum size of a minimal length closed walk which hits all vertices in a strongly connected directed graph with exactly n vertices. Clearly, $\omega(n) \leq n^2$ by composition of shortest paths $i \rightarrow i + 1$ to get $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$. On the other hand, there is an example which shows that up to a modest multiplicative factor, in general one can't do better. We thank Richard Brualdi for this example.

Example B.2. For $n = 2k \geq 4$, consider the directed graph on vertices $\{1, 2, \dots, 2k\}$ for which the set of nonzero entries of the adjacency matrix is the union of the following sets:

$$\begin{aligned} & \{(1, j) : 1 \leq j \leq k\}, \quad \{(i, k + 1) : 1 \leq j \leq k\}, \\ & \{(i, i + 1) : k \leq i < 2k\}, \quad \{(2k, 1)\}. \end{aligned}$$

For this directed graph, $\omega(2k) \geq (k - 1)(k + 2)$ and therefore $\omega(n) \geq \frac{1}{4}n^2$.

Proposition B.3. *Suppose \mathcal{U} is a unital nondiscrete subring of \mathbb{R} and A is an $n \times n$ primitive matrix over U_+ with positive trace. Then A is SSE over U_+ to a positive matrix \tilde{A} which is not larger than $2n^2 \times 2n^2$.*

Proof. The matrix operations used in the proof of Proposition B.1 give elementary SSEs over \mathcal{B} , which can be mimicked over U_+ by matrices with the same zero/nonzero pattern to give the SSE over U_+ to a positive matrix \tilde{A} . The matrix \tilde{A} is not larger than $2\omega(n) \times 2\omega(n)$, and $2\omega(n) \leq 2n^2$. \square

Lemma B.4. *Suppose \mathcal{U} is a unital nondiscrete subring of \mathbb{R} , A is an $n \times n$ primitive real matrix positive trace and \tilde{A} is a positive matrix SSE over \mathbb{R} to A and*

constructed from A using the algorithm of the proof of Proposition B.3. Suppose $\delta > 0$. Then there is $\nu > 0$ such that the following holds. If A' is a positive matrix over \mathcal{U} and $\|A' - A\| < \nu$, then A' is SSE over \mathcal{U}_+ , through positive matrices over \mathcal{U} , to a matrix \tilde{A}' such that $\|\tilde{A}' - \tilde{A}\| < \delta$.

Proof. The construction of \tilde{A} in proposition B.3 proceeds by mimicking over \mathbb{R}_+ the Boolean construction in the proof of Proposition B.1. There are three steps in the construction of \tilde{A} .

- (1) Mimicking the row splittings of the Boolean image of A to a Boolean matrix B , there are row splittings over \mathbb{R}_+ of A to some matrix B^* .
- (2) Mimicking the splitting of the $m \times m$ Boolean B to the $2m \times 2m$ Boolean C , there is a real $2m \times 2m$ matrix C^* split from B^* .
- (3) Mimicking the Boolean SSE from C to a matrix with no zero entry, there is a string of elementary matrices E_t which produces an SSE from C^* to \tilde{A} . Let ℓ be the lag of this SSE, and use a notation $C^* = M_0$, $\tilde{A} = M_\ell$, $M_t = (E_t)^{-1}M_{t-1}E_t$. For $1 \leq t \leq \ell$, there is an $\epsilon_t > 0$ which without loss of generality we assume is in \mathcal{U} , such that
 - if $1 \leq t \leq m - 1$, then $(E_t)^{-1} \geq 0$ and $M_{t-1}E_t$ is obtained from M_{t-1} by subtracting ϵ_t times column $t + 1$ from column t .
 - if $m \leq t \leq \ell$, then $E_t \geq 0$ and there are i, j such that $(E_t)^{-1}M_{t-1}$ is obtained from M_{t-1} , in which row j has no zero entry, by subtracting ϵ_t times row i from row j .

First consider step 3. Suppose M'_0 is a positive $2m \times 2m$ matrix. We recursively define $M'_t = (E_t)^{-1}M'_{t-1}E_t$, for $t = 1, 2, \dots, \ell$. Then define δ' to be the minimum of δ and the smallest positive entry in a matrix of the form $M_t, E_t, (E_t)^{-1}M_{t-1}$ or $M_{t-1}E_t, 1 \leq t \leq \ell$. Suppose the following hold:

- (i) M'_0 is close enough to M_0 that for $1 \leq t \leq \ell$, if G is a matrix in one of the four forms above, and G' is defined by replacing M_t with M'_t wherever it appears in the definition of G , then $\|G' - G\|_{\max} < \delta'$.
- (ii) For $1 \leq t < m$ and $1 \leq i \leq 2m$, if $C(i, t) = C(i, t + 1) = 0$, then $M'_0(i, t + 1) < (1/\epsilon_t)M'_0(i, t)$.

We claim that the matrices M'_t are then positive, and for $1 \leq t \leq \ell$, there is an ESSE over \mathcal{U}_+ from M'_{t-1} to M'_t . For $1 \leq t < m - 1$, the conditions (i) and (ii) imply that $M'_{t-1}E_t \geq 0$ and $M'_t > 0$, and the matrices $(E_t)^{-1}$ and $M'_{t-1}E_t$ give the required ESSE over \mathcal{U}_+ . For $t \geq m$, condition (i) implies the matrices $(E_t)^{-1}M'_{t-1}$ will be nonnegative, and again we get the ESSE over \mathcal{U}_+ . It also follows from (i) that $\|M'_\ell - \tilde{A}\|_{\max} < \delta$.

To finish, we first note that by taking ν sufficiently small we can approximate the splittings from A to B^* in the first step arbitrarily closely by splittings from A' to a matrix B'^* through positive matrices over \mathcal{U} ; and for the second step, given a positive matrix over \mathcal{U} close to B , we can split B'^* to a positive matrix $M'_0 = \begin{pmatrix} P' & B' \\ P' & B' \end{pmatrix}$ over \mathcal{U} close to C^* and also satisfying the inequalities listed in condition (ii). \square

APPENDIX C. POSITIVE INVARIANT TETRAHEDRA

Suppose A and B are positive $n \times n$ matrices over \mathbb{R} , and are conjugate over \mathbb{R} . In this section we describe the approach introduced in [14] for finding a positive path A_t of conjugate matrices from A to B . Without loss of generality, we suppose that A and B have spectral radius 1.

First we give some terminology from [14]. A *positive tetrahedron* is an n -tuple τ of vectors in an $n - 1$ dimensional real vector space such that the origin is contained in the interior of its convex hull $C(\tau)$. With respect to a given linear endomorphism T , an *invariant positive tetrahedron* is a positive tetrahedron τ such that $C(\tau)$ is mapped into its interior by T .

Now, suppose that $(A_t) = (G_t^{-1}AG_t)$ is a path of positive matrices from $A = A_0$ to $B = A_1$. We can deform such a path to a path of positive stochastic matrices, so we assume now that the A_t are positive and stochastic (so, letting r denote the column vector with every entry 1, we have $A_t r = r$). Let e_i denote the row vector which is the i th canonical basis vector. Let ℓ^t be the stochastic left eigenvector of A_t and set $v_i^t = e_i - \ell^t$, the projection of e_i along $\mathbb{R}r$ to the A_t invariant subspace W of vectors whose entries sum to zero. The tuple (v_1^t, \dots, v_n^t) has convex hull which contains the origin in its interior. Set $w_i^t = v_i^t G_t^{-1}$. Then $\tau_t = (w_1^t, \dots, w_n^t)$ is a positive invariant tetrahedron with respect to the linear transformation $T : W \rightarrow W$ defined by $w \mapsto wA$.

The path (A_t) gives rise to the path τ_t . Given t , the matrix A_t is recovered from the action of A on τ_t , as follows. For each i , the vector $w_i^t A$ is a unique convex combination of the w_j^t (which are the extreme points of $C(\tau_t)$), and the coefficients for this convex combination are provided by row i of the matrix A_t , as follows:

$$\begin{aligned} w_i^t A &= (v_i^t G_t^{-1})(G_t A_t G_t^{-1}) = v_i^t A_t G_t^{-1} = \sum_j A_{ij}^t v_j^t G_t^{-1} \\ &= \sum_j A_{ij}^t w_j^t. \end{aligned}$$

Conversely, starting from a path of positive invariant tetrahedra from τ_0 to τ_1 , we have a path (A_t) of positive stochastic matrices, with the A_t defined as above. Given t , there is a unique matrix G_t such that $v_i^0 G_t = v_i^t$ for $1 \leq i \leq n$ and $G_t r = r$, and for this matrix we have $A_t = G_t^{-1} A G_t$.

Now, to find a path of positive invariant tetrahedra, one passes (for example, see Lemmas 6.2, 6.3) to considering a path of matrices $A_t = G_t^{-1} A G_t$ with $A_t r = r$ for every t . As before, define the vectors v_i^t and w_i^t to get a path of positive tetrahedra τ_t . Now, if A_t is not positive, then τ_t will not be an invariant positive tetrahedron. The problem of deforming the path (A_t) to a path of conjugate positive matrices is replaced with the problem of deforming the path (τ_t) to a path of invariant positive tetrahedra. So, one is led to study the set of connected components of invariant positive tetrahedra for a positive matrix.

There is more information about these components in the thesis [6] of Chuy-surichay.

APPENDIX D. A LOCAL CONNECTEDNESS CONDITION FOR NILPOTENT MATRICES

Recall, $\|C\|_{\max}$ denotes the maximum absolute value of an entry of C . This is the norm we use through this appendix. For convenient reference, we repeat two definitions.

Definition 7.1. For an $n \times n$ real matrix A , and $\epsilon > 0$, $\mathcal{N}_\epsilon(A)$ denotes the set of $n \times n$ matrices B such that $\|B - A\|_{\max} < \epsilon$.

Definition 7.7. Suppose N is an $n \times n$ nilpotent matrix and \mathcal{C} is a conjugacy

class of $n \times n$ nilpotent matrices. We say \mathcal{C} is *locally connected at N* if for every $\epsilon > 0$ there exists $\delta > 0$ such that any two matrices in $\mathcal{C} \cap \mathcal{N}_\delta(N)$ are connected by a path in $\mathcal{C} \cap \mathcal{N}_\epsilon(N)$.

We introduce some notation. I_t is the $t \times t$ identity matrix, 0_t is the $t \times t$ zero matrix, and e_i denotes a zero-one row vector whose only nonzero entry is in coordinate i . The direct sum of square matrices A, B is the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. J_n is the $n \times n$ Jordan block matrix: the $n \times n$ zero-one matrix J such that $J(i, j) = 1$ iff $1 \leq i < j \leq n$ and $j = i + 1$. A matrix in Jordan form is a direct sum of Jordan blocks; the matrix has a zero Jordan block iff its kernel is not contained in its image.

Definition D.1. Suppose M is a nilpotent matrix in Jordan form. Then

- $h(M)$ is the maximum size of a Jordan block summand of M .
- $\beta(M)$ is the number of Jordan block summands of M of size at least 2×2 .
- $\beta^{\text{top}}(M)$ is the number of Jordan block summands of M of size $h(M) \times h(M)$.

For N nilpotent in a conjugacy class \mathcal{C} , we define $h(N)$ and $h(\mathcal{C})$ to be $h(M)$ for any M in Jordan form conjugate to N . Similarly for β and β^{top} .

Theorem D.2. *Suppose N is a nilpotent $n \times n$ matrix and \mathcal{C} is a conjugacy class of nilpotent $n \times n$ matrices, such that the following hold:*

- (1) *The Jordan form of a matrix in \mathcal{C} has a zero block.*
- (2) *$h(\mathcal{C}) \geq h(N)$.*
- (3) *$\beta^{\text{top}}(\mathcal{C}) \geq \beta(N)$.*

Then \mathcal{C} is locally connected at N .

The necessity of condition (2) in the statement is clear. Without condition (1), N can be the limit of matrices from different connected components of \mathcal{C} , as happens with $t > 0$ in the following example:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Question D.3. Does Theorem D.2 remain true if the assumption (3) is removed?

Our partial result and the structure of the nilpotent matrices as a stratified space [26, 27, 29] suggest the answer may be yes.

It is clear that the theorem holds for N if and only if it holds for some matrix conjugate to N . For the proof, we will make explicit constructions using a matrix of a specific form. We will formulate the constructive result below as a technical lemma, for which we make some preparations.

Theorem D.2 is true if $N \in \mathcal{C}$ (Lemma 5.1) and it is vacuously true for \mathcal{C} if N is not a limit of matrices from \mathcal{C} . So, we assume from here that $N \notin \mathcal{C}$ and N is a limit of matrices from \mathcal{C} , which implies for M in \mathcal{C} that $\text{rank}(M^k) \geq \text{rank}(N^k)$, $0 \leq k \leq n$. If $N = 0$, then Theorem D.2 can be proved quickly with the argument of Step 2 of Stage 4 below. So we also assume from here that $N \neq 0$, which means $\beta(N) \geq 1$. Set $\beta = \beta(N)$ and $h = h(\mathcal{C})$.

Given k with $1 \leq k < h$, we define the $h \times h$ matrix N_k by the rule

$$\begin{aligned} N_k(i, j) &= 1 \quad \text{if } 1 \leq i \leq k \text{ and } j = i + 1 \\ &= 0 \quad \text{otherwise .} \end{aligned}$$

The first k rows of N_k equal those of J_h and the remaining rows of N_k are zero.

We also fix a list k_1, \dots, k_β with $k_i \geq 2$ for each i , such that N is conjugate to the direct sum of $J_{k_1} \oplus J_{k_2} \oplus \dots \oplus J_{k_\beta}$ and a zero matrix. Then we fix the form we will use for our matrix N :

$$N = N_{k_1} \oplus N_{k_2} \oplus \dots \oplus N_{k_\beta} \oplus 0_{n-\beta h}$$

and define some associated subsets of $\{1, 2, \dots, n\}$,

$$\begin{aligned} \mathcal{J} &= \{i : \text{row } i \text{ of } N \text{ is nonzero}\} \\ \mathcal{K} &= \{i : \text{row } i \text{ of } N \text{ is zero}\} \\ \mathcal{T} &= \{1 + rh : 0 \leq r < \beta\} . \end{aligned}$$

Definition D.4. Given $M = J_h \oplus J_h \oplus \dots \oplus J_h$, let J be an $(n - \beta h) \times (n - \beta h)$ matrix in Jordan form such that $\begin{pmatrix} M & 0 \\ 0 & J \end{pmatrix}$ is in \mathcal{C} and has row n and column n zero.

(The condition that the n th row and column of C can be chosen zero is possible by the condition (1) in Theorem D.2.) The set \mathcal{T} indexes the rows of M through the top rows of its first β Jordan blocks, each of which is J_h . These are also the top rows of the diagonal blocks N_{k_i} in N .

Given $\epsilon > 0$, define $M(\epsilon)$ to be the $n \times n$ matrix such that

$$\begin{aligned} M(\epsilon)(i, j) &= N(i, j) \quad \text{if } i \in \mathcal{J} \\ &= \epsilon M(i, j) \quad \text{if } i \notin \mathcal{J} . \end{aligned}$$

Given $\delta > 0$, \mathcal{M}_δ denotes the set of $n \times n$ matrices C such that the following hold:

- (i) C is conjugate to M .
- (ii) $\|C - N\|_{\max} < \delta$.

We say $C \in \mathcal{M}_\delta^0$ if $C \in \mathcal{M}_\delta$ and in addition

- (iii) If $i \in \mathcal{J}$, then row i of C equals row i of N .

We will use \mathcal{M} and \mathcal{M}^0 to denote the union over $\delta > 0$ of \mathcal{M}_δ and \mathcal{M}_δ^0 (respectively).

Example D.5. For the matrix arguments to follow, it may be helpful to have the block structure of an example in view. For this example, we take

$$\begin{aligned} h &= 5 , \quad \beta = 2 , \quad n = 12 , \quad k_1 = 2 , \quad k_2 = 1 , \\ M &= J_5 \oplus J_5 \oplus 0_2 , \\ N &= N_2 \oplus N_1 \oplus 0_2 . \end{aligned}$$

Now a matrix C in \mathcal{M}_δ^0 has a block structure:

$$C = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$$

in which each \bullet has absolute value less than δ . If C is only in \mathcal{M}_δ , then the entries marked 0 and 1 above are only approximated to within δ . Continuing the example, we have

$$M(\epsilon) = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The example is somewhat special in that the summand 0_2 of M could have been much more complicated. However, it turns out that this possible complication doesn't matter in the proof below until the last stage, where it is not a big problem.

We are finally ready to state the technical lemma, from which Theorem D.2 follows immediately.

Lemma D.6. *Given $0 < \gamma < 1/49$, there exists $\delta > 0$ such that for all C in \mathcal{M}_δ and all ϵ such that $0 < \epsilon < \delta$, there is a path in \mathcal{M}_γ from C to $M(\epsilon)$.*

Proof. The path will be a concatenation of paths constructed in four stages. Combining the estimates, given $0 < \gamma < 1/49$, the lemma will hold for

$$\delta = \frac{n^{1/2^n} \gamma^{2^n}}{4(2n+2)^n}.$$

We do not claim this estimate or the requirement $\gamma < 1/49$ are sharp.

Below, subscripted matrices C in different stages are dummy variables not related to subscripted matrices C in other stages.

Stage 1. Given C in \mathcal{M}_μ , we produce a path in $\mathcal{M}_{\kappa\mu}$ to a matrix C_S in $\mathcal{M}_{\kappa\mu}^0$, where $\kappa = [2(n+1)]^S$ and $S = \#\mathcal{J} < n$. For this stage, let $i_1 < i_2 < \dots < i_S$ denote the elements of \mathcal{J} . Set $C_0 = C$. For $1 \leq s \leq S$, given C_{s-1} , we will define inductively C_s and a path $(\widetilde{C}_t)_{0 \leq t \leq 1}$ from $C_{s-1} = \widetilde{C}_0$ to $C_s = \widetilde{C}_1$ such that there is a $\kappa > 0$, independent of C , such that the following hold whenever $0 < \mu < 1/2$.

- (1) For $i = i_s$, $e_i \widetilde{C}_1 = e_{i+1}$.
- (2) For $i = i_t < i_s$, $e_i \widetilde{C}_1 = e_{i+1}$.
- (3) If $C_{s-1} \in \mathcal{M}_\mu$, then $\widetilde{C}_t \in \mathcal{M}_{\kappa\mu}$, $0 \leq t \leq 1$.

(Then the rows i_1, i_2, \dots, i_s of C_s will equal the corresponding rows in N .) The path will be a (renormalized) concatenation of two paths. The first path $(C'_t)_{0 \leq t \leq 1}$ moves the $(i_s, i_s + 1)$ entry of C_{s-1} to $1 = N(i_s, i_s + 1)$. Let $\eta = C_{s-1}(i_s, i_s + 1)$. For $0 \leq t \leq 1$, define $C'_t = D_t C_{s-1} D_t^{-1}$, where D_t is diagonal and equal to I except at $D_t(i_s + 1, i_s + 1) = \eta^t$. (Note, given $\mu < 1$, we have $\eta > 0$.) For each t , the rows i_1, i_2, \dots, i_{s-1} of C'_t equal those of N , by the induction hypothesis. At the two types of entry where $C'_t(i, j)$ might not equal $N(i, j)$, we have the following.

- If $j \neq i_s + 1$ and $i = i_s + 1$, then $C'_t(i, j) = \eta^t C_{s-1}(i, j)$.
- If $i \neq i_s + 1$ and $j = i_s + 1$, then $C'_t(i, j) = \eta^{-t} C_{s-1}(i, j)$.

In both of these cases, if $(i, j) \neq (i_s, i_s + 1)$, then $N(i, j) = 0$ and $|C_{s-1}(i, j)| < \mu$. Also, because $1/2 < \eta < 3/2$, for $|t| \leq 1$ we have $\eta^t < 2$, and consequently in both cases $|N(i, j) - C'_t(i, j)| = |C'_t(i, j)| < 2\mu$. Lastly, as t moves from 0 to 1, $C'_t(i_s, i_s + 1)$ moves monotonically from η to 1. We conclude $(C'_t)_{0 \leq t \leq 1}$ is a path in $\mathcal{M}_{2\mu}$.

We now replace C_{s-1} with C'_1 , and for notational simplicity denote it as C . For $0 \leq t \leq 1$, define an $n \times n$ matrix V_t by setting

$$\begin{aligned} V_t(i_s + 1, i) &= -tC(i_s, i) && \text{if } i \neq i_s + 1 \\ V_t(i, j) &= I(i, j) && \text{otherwise.} \end{aligned}$$

Define $\widetilde{C}_t = V_t^{-1} C V_t$. Then V_1 acts to add multiples of column $i_s + 1$ of C to other columns so that row i_s of $C V_1$ equals e_{i_s+1} . The rows of \widetilde{C}_t and $C V_t$ must be equal, except for row $i_s + 1$. It follows that (1) and (2) hold. Also,

$$\begin{aligned} \|N - V_t^{-1} C V_t\| &\leq \|N - C\| + \|C - C V_t\| + \|C V_t - V_t^{-1} C V_t\| \\ &\leq \mu + \mu^2 + (n-1)\mu < (n+1)\mu. \end{aligned}$$

So, combining this path together with the diagonal conjugation path, property (3) holds with $\kappa = 2(n+1)$. We now pass from C_0 to C_S . This completes the proof for Stage 1.

Stage 2. Given C_0 in \mathcal{M}_μ^0 , with $\mu > 0$, we produce a path in \mathcal{M}_μ^0 to a matrix M satisfying the following condition:

- (iv) If $i \in \mathcal{T}$, then $e_i M^{h-1} \neq 0$.

So, suppose $C_0 \in \mathcal{M}_\mu^0$, and let $\mathcal{U} = \{i : e_i C_0^{h-1} \neq 0\}$. Suppose $\mathcal{T} \not\subset \mathcal{U}$. We have $\#\mathcal{U} \geq \#\mathcal{T}$, because $\beta^{\text{top}}(C_0) \geq \beta(N) = \beta = \#\mathcal{T}$. Therefore, we can choose an injection $\mathcal{T} \setminus (\mathcal{T} \cap \mathcal{U}) \rightarrow \mathcal{U} \setminus (\mathcal{T} \cap \mathcal{U})$, $i \mapsto \xi(i)$. Let $\{i_1, \dots, i_R\}$ now denote the set $\mathcal{T} \setminus (\mathcal{T} \cap \mathcal{U})$. We will define matrices C_1, \dots, C_R inductively. For $1 \leq r \leq R$, given C_{r-1} we will define a path \widetilde{C}_t , $0 \leq t \leq \nu$, such that $\widetilde{C}_0 = C_{r-1}$ and such that the following hold for each t .

- (1) For $t \neq 0$ and $i = i_r$, $e_i(\widetilde{C}_t)^{h-1} \neq 0$.

- (2) If $i \in \mathcal{T}$ and $e_i(C_{r-1})^{h-1} \neq 0$, then $e_i(\widetilde{C}_t)^{h-1} \neq 0$.
(3) If $i \in \mathcal{J}$, then $e_i \widetilde{C}_t = e_{i+1}$
(4) $\|\widetilde{C}_t - C_{r-1}\|_{\max} < \mu - \|C_{r-1} - N\|_{\max}$.

(The conditions (3) and (4) keep the path in \mathcal{M}_μ^0 .) We then define the matrix M of (iv) to be C_R .

So, to define the path, suppose we are given C_{r-1} . For notational simplicity, we let C denote C_{r-1} ; j denote $\xi(i_r)$; k be the k_i such that row i_r is the top row of N_{k_i} ; and let i_r be 1. Because e_i is in the image of C if $1 < i \leq k$, we have $j \notin \{1, \dots, k\}$. Given a scalar t , let V_t be the $n \times n$ matrix such that

$$\begin{aligned} \text{row } i \text{ of } V_t &= e_i + te_j C^{i-1}, & 1 \leq i \leq k+1, \\ &= e_i, & \text{otherwise.} \end{aligned}$$

We keep t small enough that V_t is invertible, and define

$$\widetilde{C}_t = V_t C V_t^{-1}.$$

Now we verify the induction conditions. The proof of (1) is a computation:

$$\begin{aligned} e_1(V_t C V_t^{-1})^{h-1} &= e_1 V_t (C^{h-1} V_t^{-1}) = (e_1 + te_j)(C^{h-1} V_t^{-1}) \\ &= te_j C^{h-1} V_t^{-1} \neq 0. \end{aligned}$$

The proof for (2) is similar. If $i \in \mathcal{T}$ and $e_i C^{h-1} \neq 0$, then

$$e_i(V_t C V_t^{-1})^{h-1} = e_i V_t (C^{h-1} V_t^{-1}) = e_i (C^{h-1} V_t^{-1}) \neq 0.$$

For (3), given $i \in \mathcal{J}$, we must show that $e_i \widetilde{C}_t = e_{i+1}$. We do this for two cases. If $1 \leq i \leq k$, then

$$\begin{aligned} e_i(V_t C V_t^{-1}) &= (e_i V_t)(C V_t^{-1}) = (e_i + te_j C^{i-1})(C V_t^{-1}) \\ &= (e_{i+1} + te_j C^i) V_t^{-1} = e_{i+1}. \end{aligned}$$

If $i \in \mathcal{J} \setminus \{1, \dots, k\}$, then $\{i, i+1\} \cap \{1, \dots, k\} = \emptyset$; so, if $e_i C = e_{i+1}$, then

$$e_i V_t C V_t^{-1} = e_i C V_t^{-1} = e_{i+1} V_t^{-1} = e_{i+1}.$$

It is clear that (4) holds if ν is sufficiently small. This completes the proof for Stage 2.

Stage 3. We begin with C in \mathcal{M}_μ^0 , with $0 < \mu < 1/49$, with C (from Stage 2) the matrix M satisfying condition (iv) of Stage 2. We produce a path in \mathcal{M}_ν^0 from C to a matrix C_G whose first βh rows have the form of $M(\epsilon)$, but with the ϵ 's replaced perhaps by various positive numbers, with $\nu = (2n)^2 \mu^{1/2^n}$. Then, given $0 < \epsilon < 1$, we will have $C_G \in \mathcal{M}_\epsilon^0$ if

$$\mu < \frac{n^{(1/2^n)} \epsilon^{2^n}}{4}.$$

For the proof, we will inductively produce a finite sequence of matrices C_g and index sets \mathcal{S}_g , $0 \leq g \leq G$, with $G < \beta h$, beginning with $C_0 = C$. Property (iv) from Stage 2 will be preserved at every step, because successive matrices will be conjugate by a conjugacy respecting the subspaces $\mathbb{R}e_i, i \in \mathcal{T}$.

Given M in \mathcal{M}^0 , define the set $\mathcal{S}(M)$ to be the largest subset \mathcal{S} of $\{1, 2, \dots, \beta h\}$ satisfying the following conditions:

(A1) If $i \in \mathcal{S}$, then

$$\begin{aligned} \text{row } i \text{ of } M &= e_{i+1} && \text{if } i \in \mathcal{J} \\ &= 0 && \text{if } h \text{ divides } i \\ &= \text{a positive multiple of } e_{i+1}, && \text{otherwise.} \end{aligned}$$

(A2) If $0 \leq r < \beta$ and $1 \leq j \leq h$ and $rh + j \in \mathcal{S}$, then \mathcal{S} contains $\{rh + i : 1 \leq i \leq j\}$.

Note, $\mathcal{S}(M)$ contains \mathcal{J} . Also define

$$\begin{aligned} \mathcal{R}(M) &= \{i : i \leq \beta h, i \notin \mathcal{S}(M), i-1 \in \mathcal{S}(M)\} \\ \mathcal{P}(M) &= \{(i, j) : i \in \mathcal{R}(M), j \notin \mathcal{S}(M) \cup \mathcal{R}(M), M(i, j) \neq 0\}. \end{aligned}$$

Let $\mathcal{S}_g = \mathcal{S}(C_g)$, $\mathcal{R}_g = \mathcal{R}(C_g)$, $\mathcal{P}_g = \mathcal{P}(C_g)$. Continuing Example D.5, with $\mathcal{S}_g = \{1, 2, 3, 6, 7\}$ we would see the matrix C_g having the following form:

$$\begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet \\ \bullet & \bullet \\ \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_2 & 0 & 0 & 0 & 0 \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$$

In this example, $\mathcal{R}_g = \{4, 8\}$; $(i, j) \in \mathcal{P}_g$ iff $i \in \{4, 8\}$ and $j \in \{5, 9, 10, 11, 12\}$.

We will arrange by induction that the following hold for $g \geq 1$.

(B1) If $\#\mathcal{S}_{g-1} \neq \beta h$, then \mathcal{S}_{g-1} is properly contained in \mathcal{S}_g .

(B2) If $C_{g-1} \in \mathcal{M}_\mu^0$, with $\mu < 1$, then there is a path in $\mathcal{M}_{2n\sqrt{\mu}}^0$ from C_{g-1} to C_g .

Given all this, we define G to be the index g at which $\mathcal{S}_g = \{1, 2, \dots, \beta h\}$.

Now, suppose we are given C_{g-1} and \mathcal{S}_{g-1} with $\#\mathcal{S}_{g-1} < \beta h$ (i.e., \mathcal{R}_{g-1} is nonempty). We will show \mathcal{P}_{g-1} is nonempty. Pick $i \in \mathcal{R}_{g-1}$.

If i is divisible by h , then (by property A2) let t in \mathcal{T} be such that row i of C_{g-1} (which is $e_i C_{g-1}$) is a positive multiple of $e_t C^h$, and therefore is zero. Since $i-1 \in \mathcal{S}_{g-1}$, it follows that $i \in \mathcal{S}_{g-1}$, a contradiction. So i is not divisible by h . Let k be the positive integer in $[1, h-1]$ such that $e_i C_{g-1} = e_t C_{g-1}^k$.

Now suppose $C_{g-1}(i, j) \neq 0$ implies $j \in \mathcal{S}_{g-1} \cup \mathcal{R}_{g-1}$. Then $e_i C_{g-1}$ is a linear combination of the vectors $e_\tau C^j$ such that $\tau \in \mathcal{T}$ and $0 \leq k < h$. This is a contradiction, because the set $\{e_t C_{g-1}^j : t \in \mathcal{T}, 0 \leq j < h\}$ is linearly independent, by property (iv). Therefore there is a j such that $(i, j) \in \mathcal{P}_{g-1}$.

From here, we will handle the inductive transition from $g-1$ to g in three steps, given C_{g-1} and \mathcal{S}_{g-1} with $\#\mathcal{S}_{g-1} < \beta h$. By a *signed transposition matrix* for indices i, j we mean a matrix Q which is equal to the permutation matrix P for the

transposition exchanging i and j , except that one of the entries $Q(i, j)$ or $Q(j, i)$ is -1 .

STEP 1. Given $C_{g-1} \in \mathcal{M}_\mu^0$, we produce an index i in \mathcal{R}_{g-1} , and a matrix Q which is either I or is a signed transposition matrix for indices outside $\mathcal{R}_{g-1} \cup \mathcal{S}_{g-1}$, such that the following hold for the matrix $\bar{C} = Q^{-1}C_{g-1}Q$:

- (D1) $(i, i+1) \in \mathcal{P}(\bar{C})$.
- (D2) $\bar{C}(i, i+1) = \max\{|\bar{C}(i', j')| : (i', j') \in \mathcal{P}(\bar{C})\}$.
- (D3) There is a path in $\mathcal{M}_{\sqrt{\mu}}^0$ from C_{g-1} to \bar{C} .
- (D4) $\mathcal{S}(\bar{C}) = \mathcal{S}_{g-1}$, and $\|N - \bar{C}\|_{\max} = \|N - C_{g-1}\|_{\max} < \mu$.

STEP 2. For the matrix \bar{C} produced in Step 1, defining $\alpha = \|N - \bar{C}\|_{\max}$, we produce a matrix C' and $(i, i+1) \in \mathcal{P}_{g-1}$ such that the following hold.

- (C1) $\|N - C'\|_{\max} = |C'(i, i+1)| = \sqrt{\alpha}$.
- (C2) If $r \in \mathcal{S}_{g-1}$, then $|C'(i, r)| < \mu$.
- (C3) There is a path in $\mathcal{M}_{\sqrt{\mu}}^0$ from \bar{C} to C' .

STEP 3. Given C' from Step 2, we produce a path in $\mathcal{M}_{2n\sqrt{\mu}}^0$ from C' to the desired matrix C_g .

PROOF FOR STEP 1.

Choose (i, j) from the nonempty set \mathcal{P}_{g-1} such that

$$|C_{g-1}(i, j)| = \max\{|C_{g-1}(i', j')| : (i', j') \in \mathcal{P}_{g-1}\}.$$

There are two cases.

CASE 1: $j \neq i+1$. Both j and $i+1$ are outside $\mathcal{R}_{g-1} \cup \mathcal{S}_{g-1}$. Let Q denote the $n \times n$ signed transposition matrix for indices $i+1$ and j such that

$$\begin{aligned} Q(j, i+1) &= -1, & \text{if } C_{g-1}(i, j) < 0 \\ Q(i+1, j) &= -1, & \text{if } C_{g-1}(i, j) > 0. \end{aligned}$$

Set $\bar{C} = Q^{-1}C_{g-1}Q$. We have $Q \in \text{SL}(n, \mathbb{R})$, and for a path from C_{g-1} to \bar{C} we can use $U_t^{-1}C_{g-1}U_t$, $0 \leq t \leq 1$, with (U_t) a path from I to Q . E.g., for $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we may use (in the principal submatrix on coordinates $\{i+1, j\}$)

$$\begin{aligned} U_t &= \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-t^2 & t \\ -2t+t^3 & 1-t^2 \end{pmatrix}. \end{aligned}$$

Then for $0 \leq t \leq 1$,

$$\|U_t^{-1}C_{g-1}U_t\|_{\max} \leq 4\|U_t\|_{\max}\|C_{g-1}\|_{\max} < 5\|C_{g-1}\|_{\max}.$$

CASE 2: $j = i+1$. If $C_{g-1}(i, i+1) > 0$, then set $Q = I$ and $\bar{C} = C_{g-1}$. If $C_{g-1}(i, i+1) < 0$, then let W be the matrix in $\text{SL}(n, \mathbb{R})$ obtained from I_n by multiplying rows $i+1$ and n by -1 , and define $\bar{C} = W^{-1}C_{g-1}W$. As in Case 1, we may produce a path from C_{g-1} to \bar{C} by conjugating with a path (W_t) , $0 \leq t \leq 1$, from I to W . One such path is given (on the principal submatrix on

indices $\{i+1, n\}$ by

$$\begin{aligned} W_t &= \begin{pmatrix} 1 & 0 \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 2t^2 & 2t - 2t^3 \\ -4t + 4t^3 & 1 - 6t^2 + 4t^4 \end{pmatrix} \end{aligned}$$

Then

$$\|W_t^{-1}C_{g-1}W_t\|_{\max} \leq 4\|W_t\|_{\max}\|C_{g-1}\|_{\max} < 7\|C_{g-1}\|_{\max}.$$

In both cases, (D1) and (D2) hold, and the path from C_{g-1} to \bar{C} is contained in $\mathcal{M}_{7\mu}^0$, and consequently in $\mathcal{M}_{\sqrt{\mu}}^0$, since $\mu < 1/49$. This completes the proof for Step 1.

PROOF FOR STEP 2.

Given \bar{C} and $(i, i+1) \in \mathcal{P}(\bar{C})$ from Step 1, define $\beta = \max\{|\bar{C}(i, j)| : (i, j) \in \mathcal{P}(\bar{C})\}$. Define a path $C'_t = D_t \bar{C} D_t^{-1}$, $1 \leq t \leq \sqrt{\alpha}/\beta$, from $\bar{C} = C'_1$ to $C' = C'_{\sqrt{\alpha}/\beta}$, in which D_t is the diagonal matrix defined by

$$\begin{aligned} D_t(i', i') &= t, \quad \text{if } i' \in \mathcal{S}_{g-1} \cup \mathcal{R}_{g-1} \\ &= 1, \quad \text{otherwise.} \end{aligned}$$

We have $|C_{g-1}(i, r)| < \mu$ for all r , because $i \notin \mathcal{J}$. Therefore, $|\bar{C}(i, r)| < \mu$ for all r . So, if $r \in \mathcal{S}_{g-1} \cup \mathcal{R}(\bar{C})$, then $|C'_t(i, r)| \leq |\bar{C}(i, r)| < \mu$; this establishes (C2). If $i' \in \mathcal{S}_{g-1}$ and $j' \notin \mathcal{S}_{g-1} \cup \mathcal{R}(\bar{C})$, then $C_{g-1}(i', j') = 0$, and $\bar{C}(i', j') = 0$. Therefore, $|C'_t(i', j')| > |\bar{C}(i', j')|$ is possible only if $i' \in \mathcal{R}_{g-1}$ and $j' \notin \mathcal{S}_{g-1} \cup \mathcal{R}(\bar{C})$. It follows then from (D2) that $\|C'_t - N\| \leq \sqrt{\alpha}$ for all t , with (C1) holding for $C' = C'_1$. Because $\alpha < \mu$, (C'_t) is a path in $\mathcal{M}_{\sqrt{\mu}}^0$, and then (C3) follows from (D3). This finishes the argument for Step 2.

PROOF FOR STEP 3.

For a lighter notation, in this step we will write C in place of C' for the matrix satisfying (C1)-(C3) for a given $(i, i+1)$ from $\mathcal{P}(C')$. For $0 \leq t \leq 1$, we define an $n \times n$ matrix V_t equal to I outside row $i+1$. In that row, we define $V_t(i+1, i+1) = 1$ and

$$V_t(i+1, r) = -tC(i, r)/C(i, i+1), \quad \text{if } r \neq i+1.$$

Define the path $\tilde{C}_t = V_t^{-1}CV_t$, $0 \leq t \leq 1$. Then $C = \tilde{C}_0$, and we define $C_g = \tilde{C}_1$.

First we will check that C_g satisfies condition (B1). The matrix V_1 acts to add multiples of column $i+1$ of C to other columns so that row i of CV_1 has exactly one nonzero entry, which is at position $(i, i+1)$. Suppose $i' \in \mathcal{S}_{g-1}$. If $C(i', i+1) \neq 0$, then $i+1 = i'+1$, which forces $i = i' \in \mathcal{S}_{g-1}$, contradicting $(i, i+1) \in \mathcal{P}$. Therefore $C(i', i+1) = 0$. Therefore row i' of CV_t equals row i' of C . The matrix $V_t^{-1}CV_t$ can differ from CV_t only in row $i+1$, which is not in \mathcal{S}_{g-1} , since $(i, i+1) \in \mathcal{P}$. Consequently, \mathcal{S}_g contains \mathcal{S}_{g-1} and also $\{i\}$. Since $i \notin \mathcal{S}_{g-1}$, the condition (B1) is satisfied.

Now we turn to (B2). We have

$$\begin{aligned} \text{(D.1)} \quad & \|N - V_t^{-1}CV_t\|_{\max} \\ & \leq \|N - C\|_{\max} + \|C - CV_t\|_{\max} + \|CV_t - V_t^{-1}CV_t\|_{\max} \end{aligned}$$

and we will bound the three terms on the right.

We have $\|N - C\|_{\max} < \sqrt{\mu}$, since $C \in \mathcal{M}_{\sqrt{\mu}}^0$.

Because $C - CV_t = C(I - V_t)$, column r of $C - CV_t$ is zero if $r = i + 1$ and otherwise equals column $i + 1$ of C (whose entries are smaller in absolute value than $\sqrt{\mu}$, since $i \notin \mathcal{S}_{g-1} \supset \mathcal{J}$) multiplied by $-tC(i, r)/C(i, i + 1)$ (which by (C1) has absolute value at most 1). Therefore $\|C - CV_t\|_{\max} < \sqrt{\mu}$ (and then $\|N - CV_t\|_{\max} < 2\sqrt{\mu}$).

The last term in (D.1) is the maximum over (i', j') of the absolute value of

$$(CV_t - V_t^{-1}CV_t)(i', j') = \left((I - V_t^{-1})(CV_t) \right)(i', j').$$

This quantity is zero if $i' \neq i + 1$, since row $i + 1$ is the only nonzero row of $(I - V_t^{-1})$. For $i' = i + 1$, we have

$$\begin{aligned} \left((I - V_t^{-1})(CV_t) \right)(i + 1, j') &= \sum_r (I - V_t^{-1})(i + 1, r)(CV_t)(r, j') \\ &= \sum_{r: r \neq i+1} \left(\frac{-tC(i, r)}{C(i, i + 1)} \right) (CV_t)(r, j'). \end{aligned}$$

Now we bound the terms in the last sum by two cases.

CASE 1: $r \in \mathcal{J}$.

Then $(CV_t)(r, j') = V_t(r + 1, j')$ and also $r \neq i$. If $j' \neq r + 1$, then $(CV_t)(r, j') = V_t(r + 1, j') = 0$. If $j' = r + 1$, then $(CV_t)(r, r + 1) = V_t(r + 1, r + 1) = 1$. So,

$$\begin{aligned} \left| \frac{-tC(i, r)}{C(i, i + 1)} \right| (CV_t)(r, r + 1) &= \left| \frac{-tC(i, r)}{C(i, i + 1)} \right| \\ &= \left| \frac{-tC(i, r)}{\sqrt{\alpha}} \right| \quad \text{by (C1)} \\ &< \frac{\mu}{\sqrt{\alpha}} \quad \text{by (C2)} \\ &< \frac{\mu}{\sqrt{\mu}} = \sqrt{\mu} \quad \text{by (C3)}. \end{aligned}$$

CASE 2: $r \notin \mathcal{J}$.

Then $N(r, j') = 0$ and

$$\left| \frac{-tC(i, r)}{C(i, i + 1)} \right| |(CV_t)(r, j')| \leq (1) \|N - CV_t\|_{\max} < 2\sqrt{\mu}.$$

Using the estimates above to bound the third term of (D.1) by $(n - 1)2\sqrt{\mu}$, and then substituting bounds into (D.1), we get

$$\|N - V_t^{-1}CV_t\|_{\max} < \sqrt{\mu} + \sqrt{\mu} + (n - 1)2\sqrt{\mu} = 2n\sqrt{\mu}.$$

This finishes the proof for Step 3, and for Stage 3.

Stage 4.

Given $0 < \epsilon < \mu$ and a matrix C in \mathcal{M}_μ^0 with the first βh rows agreeing with those of $M(\epsilon)$ (except that the epsilons are allowed to be different positive numbers), we produce a path in \mathcal{M}_μ^0 from C to $M(\epsilon)$.

The $n \times n$ matrix C has a block form $C = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$, in which X is $\beta h \times \beta h$.

STEP 1: Defining $C_t = \begin{pmatrix} X & 0 \\ tY & Z \end{pmatrix}$, $1 \geq t \geq 0$, we show (C_t) is a path of conjugate matrices. Conjugacy is clear for $0 < t \leq 1$, where

$$C_t = \begin{pmatrix} (1/t)I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}.$$

For $1 \leq i \leq h$, let \mathcal{Z}_i denote the set of indices from $\{1, \dots, \beta h\}$ (which indexes rows and columns of X) which are congruent to $i \pmod{h}$. To check conjugacy of C_0 and C , we first note there is a conjugacy of the form

$$C' = \begin{pmatrix} X & 0 \\ Y' & Z \end{pmatrix} = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix}$$

such that column i of Y' is zero if $i \notin \mathcal{Z}_1$. This result from a composition of conjugations arising from elementary row and column operations as follows. In decreasing order for $i = h-1, h-2, \dots, 1$: for each $j \in \mathcal{Z}_i$, and for each i' in $[\beta h+1, n]$ such that position $(i', j+1)$ of the current matrix has a nonzero entry, we add a multiple of row j to row i' , and then subtract the same multiple of column i' from column j .

We will show Y' must be zero. For this, consider row vectors in the form (u, v) , where u has βh entries and v has $n - \beta h$ entries. For $1 \leq t \leq h$, let W_t be the subspace of $\mathbb{R}^{\beta h}$ spanned by the vectors e_j such that $j \in \mathcal{Z}_i$ and $1 \leq i \leq t$. Let W_0 be the trivial space $\{0\}$. Given $(0, v)$ and $k \geq 1$, define $(u^{(k)}, v^{(k)}) = (0, v)C^k$. We claim for $1 \leq k \leq h$ that

- (1) $u^{(k)} \in W_k$.
- (2) If $u^{(1)} \neq 0$, then $u^{(k)} \in W_k \setminus W_{k-1}$.

The claim is clear for $k = 1$, because column j of Y' is zero if $j \notin \mathcal{Z}_1$. Suppose $1 \leq k < h$, and the claim holds for k . Then

$$\begin{aligned} (0, v)C^{k+1} &= (u^{(k)}, v^{(k)})C = (u^{(k)}, 0)C + (0, v^{(k)})C \\ &= (u^{(k)}X, 0) + (u'', v'') \end{aligned}$$

for some $u'' \in W_1$. The claim then holds for $k+1$ because X maps $W_k \setminus W_{k-1}$ injectively into $W_{k+1} \setminus W_k$ (since $k < h$) and W_k contains W_1 .

If $Y' \neq 0$, then there must be some v for which $(0, v)C = (u^{(1)}, v^{(1)})$ with $u^{(1)} \neq 0$. It follows from the claim that $(0, v)C^h$ is nonzero. That contradicts $C^h = 0$. So, $Y' = 0$, and C_0 is conjugate to C .

STEP 2. We begin with $C = \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}$ in \mathcal{M}_μ^0 and define a conjugacy in \mathcal{M}_μ^0 to $M(\epsilon)$. Clearly, after applying a path in \mathcal{M}_μ^0 , we can suppose that X exactly equals the corresponding $\beta h \times \beta h$ block in $M(\epsilon)$, and write $M(\epsilon)$ as $\begin{pmatrix} X & 0 \\ 0 & J \end{pmatrix}$, where J is the matrix of Definition D.4. Let $m = n - \beta h$.

The conjugacy of C and $M(\epsilon)$ and the fact that J has a Jordan zero block imply that there is a matrix $U \in \mathrm{SL}(m, \mathbb{R})$ such that $U^{-1}ZU = J$. Then there is a path (U_t) in $\mathrm{SL}(m, \mathbb{R})$ from I to U , giving a path of conjugate matrices from Z to J , $Z_t = U_t^{-1}ZU_t$. Given $\delta > 0$, we can follow a path $\begin{pmatrix} X & 0 \\ 0 & sZ \end{pmatrix}$, $1 \geq s \geq \delta$, with a path $\begin{pmatrix} X & 0 \\ 0 & sZ_t \end{pmatrix}$, $1 \geq s \geq \delta$, and then a path $\begin{pmatrix} X & 0 \\ 0 & sJ \end{pmatrix}$, $\delta \leq s \leq 1$. This gives a path of conjugate matrices from C to $M(\epsilon)$, and with δ small enough, the path is contained in \mathcal{M}_μ^0 . This completes the proof of Stage 4, and the lemma. \square

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