We remark that the TPPD sofic shifts are the most interesting sofic shifts. The following conditions are equivalent for sofic T (see [9]):

(1) T is TPPD;

(2) T is intrinsically ergodic, and $\max_T$ have full support;

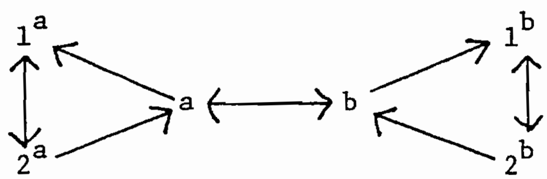
(3) T is a factor of an irreducible subshift of finite type.

From here, we suppose that T is orbit equivalent to a sofic shift S, and ask what properties T must inherit from S. First, even if S is a mixing subshift of finite type, T need not be expansive (example 4.1). If we assume that S is a MSFT and T is an expansive homeomorphism orbit equivalent to S, then we do not know (alas) if S and T must be flip conjugate; we do not know if T must be finite type or even if h(T) must equal h(S). We do have examples of orbit conjugacies between isomophic SFT's S and T which exhibit various technical pathologies. In (4.8), no conjugacy between S and T can respect the correspondence of orbits given by the orbit conjugacy. In (4.9), the set D of disoriented orbits in S is a dense G_δ with $\max_S D = 1$. We also show in (4.5) that mixing, strictly sofic orbit equivalent shifts need not be flip conjugate. In (4.6,7), we find an expansive orbit conjugate of a mixing sofic shift need not be sofic, or even retain the specification property.

For the rest of the section we suppose that S and T are sofic and orbit equivalent by bounded jumps. If S and T are not TPPD, then they need not be flip conjugate (example 4.3). For TPPD sofic shifts we do not know if orbit equivalence by bounded jumps implies flip conjugacy, but the following example shows that a flip conjugacy respecting the given correspondence of orbits is not generally possible. (In this sense, the bounded-jumps case for TPPD sofic shifts parallels the unbounded-jumps case for SFT's.)

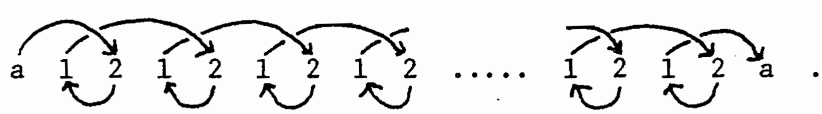
(3.2) Example

Let U be the 1-step MSFT defined by the graph



and let S be the sofic shift obtained by the 1-block which sends a to a, b to b, 1^a and 1^b to 1, 2^a and 2^b to 2. This map is one-to-one, except that it collapses the two periodic orbits $(1^a2^a)^\infty$ and $(1^b2^b)^\infty$.

Now define T orbit equivalent to S by setting $Tx = Sx$ except on blocks of the forms $a(12)^\infty$, $(12)^\infty a$, and $a(12)^n a$. There get the (bounded) jump function from the picture:



T is naturally conjugate to the sofic shift obtained from U by the 1-block map which sends a to a, b to b, 1^a and 2^b to 2, 1^b and 2^a to 1. This map is one-to-one, except that it collapses the two periodic orbits $(1^a2^a)^\infty$ and $(2^b1^b)^\infty$.

The shifts S and T are very close. Each "almost" equals their common cover U. Their restrictions to the complement of the two point orbit $(12)^\infty$ are conjugate by continuous jumps, as in (2.7). They have a common factor obtained by collapsing $(12)^\infty$ to a fixed point. However, no conjugacy between them can respect the given correspondence of orbits.

To see this, let x be a point in the orbit $(12)^\infty a(12)^\infty$. Then $\{S^{2n}x$: n is an integer$\}$ has two accumulation points. But $\{T^{2n}x$: n is an integer$\}$ has only one. □

Finally, we consider the case where T is a homeomorphism orbit conjugate to a transitive subshift of finite type S by bounded jumps. The rest of this section is devoted to showing that S and T must be flip conjugate. To that end, we recall and introduce some definitions to be used in the proof.

As in (2.7), we let X be the common domain of S and T; let P be the finite set of periodic points on whose orbits n(x) is not continuous. Then (2.7) provides a conjugacy of S and T (possibly after replacing T with T^{-1}) on the complement X~P of P, of the form $gx = T^{a(x)}x$, where $a(x) = \lim \frac{1}{2}[N_k(x) - P_k(x)]$ and a(x) is continuous/locally constant on X~P. It will be convenient to define

$\ell(x) = |\{i<0\text{: for some positive } j,\ f_j(x) = i\}|$,

the number of negative coordinates of x

used to define positive coordinates of $\bar{x}$,

and similarly to define

$r(x) = |\{i>0\text{: for some negative } j,\ f_j(x) = i\}|$,

the number of positive coordinates of x

used to define negative coordinates of $\bar{x}$.

Then it is not hard to show that

$a(x) = \ell(x) - r(x)$.

If $x_i \ldots x_j$ is periodic-B (this term was defined in lemma 2.1 of Chapter 1), but $x_{i-1} \ldots x_j$ is not, then we say that x_{i-1} is a <u>seam</u> (with respect to B). Similarly, x_{j+1} is a seam if $x_i \ldots x_{j+1}$ is not periodic-B. We call $\min\{i \geq 0\text{: } x_{k-i} \ldots x_{k+i} \text{ is not periodic-B}\}$ the distance from x_k to the nearest seam.

(3.3) <u>Theorem</u>

Suppose the following conditions hold:

(1) S is a transitive subshift of finite type of positive entropy;

(2) T is a homeomorphism with the same orbits as S;

(3) $Tx = S^{n(x)}x$, where n(x) is bounded.

Then S and T are flip conjugate by a map $x \to T^{a(x)}x$, where a(x) is continuous off a finite set of periodic points.

<u>Proof</u>

Our goal is to extend the conjugacy g given by (2.7), which we may assume (possibly after replacing T with T^{-1}) is a conjugacy of S and T on X~P, to all of X. First, suppose g extends to a continuous map g' on X. Then g'(X) is compact, so g' is surjective. Then the extension adds |P| new points to the domain and |P| new points to the range, so g' is injective. By compactness, g' is open, hence a homeomorphism. But S agrees with $(g')^{-1}Tg'$ on the dense set X~P, hence on all of X, and so g' is a conjugacy. So it is enough to show that g extends continuously to X. Suppose b is in P; let p be the period of b, and let $B = b_0 \ldots b_{p-1}$. Below, seams are defined with respect to stretches which are periodic-B; $B_k = \{x\colon x_i = b_i \text{ if } i \leq k\}$.

Since T is a homeomorphism, there exists $L > 0$ such that for any y in the orbit of b, if $y_i = x_i$ for $|i| \leq L$, then $(Ty)_i = (Tx)_i$ for $|i| \leq p$. Let

$E = \{x\colon x(0,+\infty)$ is periodic-B, and x_0 is a seam$\}$.

Each point x in E is contained in a neighborhood V_x for which there exist j, K and J for which the following hold:

(1) $\{f_i(x)\colon 0<i<j\}$ contains N consecutive integers between L and L+K (a roadblock);

(2) $J > \max\{f_i(x)\colon i \leq j\}$

(so, if y is in V_x and $I \geq J$, then $S^I y$ is in the T-future of points $S^i y$ such that y_i is in the roadblock given by (1)).

Because E iscompact, finitely many of the V_x cover E. Consequently, we may choose M such that we may make the following deduction.

Suppose we are given the following:

(1) $x \in B_M$;

(2) $f_i(x) = p$

(that is, $S^p x = T^i x$; i might be negative or positive);

(3) $0 < j < i$;

(4) $f_j(x) = k < 0$

(that is, $T^j x = S^k x$ with k negative).

Then we can conclude that $(T^j x)_{-L} \ldots (T^j x)_L = x_{k-L} \ldots x_{k+L}$ is periodic-B. By making the same argument with respect to seams on the right, we may choose M so that our deduction is valid without the assumption in (4) that k is negative. Then the choice of L forces $S^p x = T^{rp} x$ for some integer r.

We claim that if $x \in B_M$, then $a(x) \equiv a(S^p x) \bmod p$. For notational convenience, let $S^p x = T^{rp} x$ for $r > 0$; the argument for $r < 0$ is essentially the same. Recall that $f_j(x) = i$ means that $T^j x = S^i x$ and $x_i = \bar{x}_j$. To compute, define the following numbers:

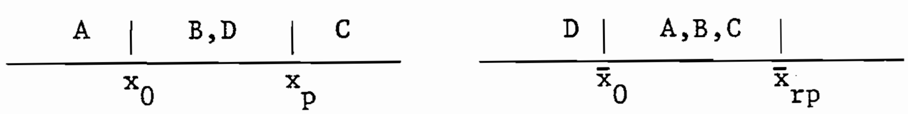
$A = |\{i\colon i < 0 \text{ and } f_j(x) = i \text{ for some } j \text{ such that } 0 < j < rp\}|;$

$B = |\{i\colon 0 < i < p \text{ and } f_j(x) = i \text{ for some } j \text{ with } 0 < j < rp\}|;$

$C = |\{i\colon i > p \text{ and } f_j(x) = i \text{ for some } j \text{ such that } 0 < j < rp\}|;$

$D = |\{i\colon 0 < i < p \text{ and } f_j(x) = i \text{ for some } j \text{ such that } j < 0\}|.$

The following picture indicates where the coordinates defining these numbers lie.



Then

$\ell(S^p x)$ = "holes to left of $(S^p x)_0$ left by the T-past of $S^p x$"

$= \ell(x) - A + (p-1) - B - D,$

$r(S^p x)$ = "spaces to right of $(S^p x)_0$ filled by the T-past of $S^p x$"

$= r(x) - D + C.$

So,

$$a(S^p x) = [\ell(S^p x)] - [r(S^p x)]$$

$$= [\ell(x) - A + (p-1) - B - D] - [r(x) - D + C]$$

$= \ell(x) - r(x) - (A + B + C) + (p-1) \equiv a(x) \bmod p$, because

$A + B + C \equiv (p-1) \mod p$.

We now have $a(x)$ well-defined mod p in long periodic stretches (the proof to this point is valid for any subshift). Roughly, the finite type assumption will let us hook up different long periodic stretches to conclude that their definitions of $a(x)$ mod p must agree. Define

$G = \{x \in X\sim P: \; x_i = b_i \; \text{ for } \; i > 0\}$,

$H = \{x \in X\sim P: \; x_i = b_i \; \text{ for } \; i < 0\}$.

Because S has a dense aperiodic orbit and X is compact, the sets $G \cap B_M$ and $H \cap B_M$ must be nonempty.

Suppose x is in $G \cap B_M$, y is in $H \cap B_M$, and the numbers $a(x)$, $a(y)$ differ modulo p. Since $a(x)$ is continuous on X~P, we may choose k greater than M such that $a(x)$ is constant on the neighborhood $\{z: z_i = x_i \text{ if } |i| \leq k\}$ of x, and on the neighborhood $\{z: z_i = y_i \; \text{ if } \; |i| \leq k\}$ of y. Pick J so that $Jp > 2k$ and S is a Jp-step subshift of finite type. Define a point w in X~P by

$w_i = x_i \;$ if $\; i \leq Jp$,

$w_i = y_{i-Jp} \;$ if $\; i \geq Jp$.

Then $a(w) = a(x)$ and $a(S^{Jp}w) = a(y)$. But since $S^{ip}w \in B_M$ for $0 \leq i \leq k$, we have $a(w) \equiv a(S^p w) \equiv \ldots \equiv a(S^{kp}w) \mod p$, a contradiction. So, $a(x)$ has the same value mod p, say α, on x and y. If $x^1, x^2 \in G \cap B_M$ and $y \in H \cap B_M$, then the same argument shows that $a(x^1)$ and $a(x^2)$ must agree mod p with $a(y)$; so, $a(x^1) \equiv a(x^2) \mod p$. Likewise, $a(x)$ is constant mod p on $H \cap B_M$; so, $a(x) \equiv \alpha$ on $(G \cup H) \cap B_M$.

Finally, suppose that a sequence $\{x^n\}$ converges to b, that the sequence $\{x^n\}$ is contained in the complement of the set $P \cup G \cup H$, and for each n $a(x^n) \equiv \beta \mod p$, where β differs from α modulo p. After discarding those elements of the sequence which are not in B_M, we can find nonpositive integers r(n) such that the following hold:

(1) r(n) is divisible by p, for each n;

(2) $(x^n)_{r(n)-M} \dots (x^n)_0$ is periodic-B, for each n;

(3) $(x^n)_{r(n)-M-p} \dots (x^n)_{r(n)-M}$ contains a seam, for each n;

(4) $\lim_n r(n) = -\infty$;

(5) if we define $y^n = S^{r(n)}(x^n)$,

then $y^n \in B_M$ and $a(y^n) \equiv a(x^n) \mod p$, for each n.

Now y^n has a subsequence converging to some point of $H \cap B_M$. This contradicts the continuity of a(x) on X~P and finishes the proof. □

4. Examples

In this section, we give ten examples, which we list below. In each of the examples, S and T denote transitive orbit equivalent homeomorphisms on a compact metric space; a statement about n(x) implies that S and T have the same orbits and $Tx = S^{n(x)}x$. The examples are largely variations on the theme of example 1, and proofs of some of the examples are given only as sketches. Example 2 shows there can be many maps orbit conjugate to a given homeomorphism. Examples 1, 3-7 are counterexamples; each limits the implications of orbit conjugacy. Examples 8-10 show ways in which the orbit conjugacy can be bad. In 8 and 9, n(x) is discontinuous at only one point and S and T are isomorphic, but the orbit conjugacy still exhibits pathology.

(4.0) List of Examples

1. S is the 2-shift, T is not expansive;
   n(x) is discontinuous at just one point.

2. Uncountably many distinct homeomorphisms share orbits with the 3-shift.

3. S and T are sofic, $n(x)$ is bounded,
   S and T are not flip conjugate.
   (Note: S and T do not have dense periodic points.)

4. S and T are mixing subshifts, $n(x)$ is bounded,
   S and T are not flip conjugate.

5. S and T are mixing sofic shifts,
   S and T are not flip conjugate.

6. S is a mixing sofic shift,
   T is a mixing shift which is not sofic.

7. S is a mixing shift with specification,
   T is a mixing shift without specification.

8. S is the 2-shift, T is isomorphic to S,
   $n(x)$ is discontinuous at just one point,
   a bilaterally transitive orbit of S may be
   a unilaterally transitive orbit of T.
   (So, no conjugacy of S and T can respect the given correspondence of orbits.)

9. S is the 2-shift, T is isomorphic to S,
   $n(x)$ is discontinuous at just one point.
   Let D be the set of disoriented orbits (made precise in 9).