STRONG SHIFT EQUIVALENCE AND THE GENERALIZED SPECTRAL CONJECTURE FOR NONNEGATIVE MATRICES

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Dedicated to Hans Schneider, in memoriam

Abstract. Given matrices A and B shift equivalent over a dense subring $\mathcal{R}$ of $\mathbb{R}$, with A primitive, we show that B is strong shift equivalent over $\mathcal{R}$ to a primitive matrix. This result shows that the weak form of the Generalized Spectral Conjecture for primitive matrices implies the strong form. The foundation of this work is the recent result that for any ring $\mathcal{R}$, the group $\mathrm{NK}_1(\mathcal{R})$ of algebraic K-theory classifies the refinement of shift equivalence by strong shift equivalence for matrices over $\mathcal{R}$.

Contents

1. Introduction

The purpose of this paper [1] is to prove the following theorem and explain its context.

Theorem 1.1. *Suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$, A is a primitive matrix over $\mathcal{R}$ and B is a matrix over $\mathcal{R}$ which is shift equivalent over $\mathcal{R}$ to A.*

Then B is strong shift equivalent over $\mathcal{R}$ to a primitive matrix.

We begin with the context. By ring, we mean a ring with 1; by a semiring, we mean a semiring containing $\{0, 1\}$. A primitive matrix is a square matrix which is nonnegative (meaning entrywise nonnegative) such that for some $k > 0$ its kth power is a positive

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[1]This paper is an outgrowth of the paper [**?**], for which its authors were awarded (along with Robert Thompson) the second Hans Schneider Prize.

matrix. Definitions and more background for shift equivalence (SE) and strong shift equivalence (SSE) are given in Section 2.

We recall the Spectral Conjecture for primitive matrices from [?]. In the statement, $\Delta = (d_1, \dots, d_k)$ is a k-tuple of nonzero complex numbers. Δ is the *nonzero spectrum* of a matrix A if A has characteristic polynomial of the form $\chi_A(t) = t^m \prod_{1\le i\le k}(t-d_i)$. Δ has a *Perron value* if there exists i such that $d_i > |d_j|$ when $j \neq i$. The *trace* of Δ is $\mathrm{tr}(\Delta) = d_1 + \cdots + d_k$. Δ^n denotes $((d_1)^n, \dots, (d_k)^n)$, the tuple of nth powers; and the *nth net trace* of Δ is

$$\mathrm{tr}_n(\Delta) = \sum_{d|n} \mu(n/d)\mathrm{tr}(\Delta^d)$$

in which μ is the Möbius function ($\mu(1) = 1$; $\mu(n) = (-1)^r$ if n is the product of r distinct primes; $\mu(n) = 0$ if n is divisible by the square of a prime).

Spectral Conjecture 1.2. [?] *Let $\mathcal{R}$ be a subring of $\mathbb{R}$. Then Δ is the nonzero spectrum of some primitive matrix over $\mathcal{R}$ if and only if the following conditions hold:*

(1) Δ has a Perron value.
(2) The coefficients of the polynomial $\prod_{i=1}^k (t-d_i)$ lie in $\mathcal{R}$.
(3) If $\mathcal{R} = \mathbb{Z}$, then for all positive integers n, $\mathrm{tr}_n(\Delta) \ge 0$;
if $\mathcal{R} \neq \mathbb{Z}$, then for all positive integers n and k,
(i) $\mathrm{tr}(\Delta^n) \ge 0$ and (ii) $\mathrm{tr}(\Delta^n) > 0$ implies $\mathrm{tr}(\Delta^{nk}) > 0$.

It is not difficult to check that the nonzero spectrum of a primitive matrix satisfies the three conditions [?]. (We remark, following [?] it is known that the nonzero spectra of symmetric primitive matrices cannot possibly have such a simple characterization.)

To understand the possible spectra of nonnegative matrices is a classical problem of linear algebra (for early background see e.g. [?]) on which interesting progress continues (see e.g. [?, ?, ?, ?] and their references). Understanding the nonzero spectra of primitive matrices is a variant of this problem and also an approach to it: to know the minimal size of a primitive matrix with a prescribed nonzero spectrum is to solve the classical problem (for details, see [?]); and it is in the primitive case that the Perron-Frobenius constraints manifest most simply.

Finally, as the spectra of matrices over various subrings of $\mathbb{R}$ appear in applications, in which the nonzero part of the spectrum is sometimes the relevant part [?, ?], it is natural to consider the nonzero spectra of matrices over arbitrary subrings of $\mathbb{R}$.

The Spectral Conjecture has been proved in enough cases that it seems almost certain to be true in general. For example, it is true under any of the following conditions:

- The Perron value of Λ is in $\mathcal{R}$ (this always holds when $\mathcal{R} = \mathbb{R}$) or is a quadratic integer over $\mathcal{R}$ [?].
- $\mathrm{tr}(\Lambda) > 0$ [?, Appendix 4]

- $\mathcal{R} = \mathbb{Z}$ or $\mathbb{Q}$ [**?**].

The general proofs in [**?**] do not give even remotely effective general bounds on the size of a primitive matrix realizing a given nonzero spectrum. The methods used in [**?**] for the case $\mathcal{R} = \mathbb{Z}$ are much more tractable but still very complicated. However, there is now an elegant construction of Tom Laffey [**?**] which proves the conjecture for $\mathcal{R} = \mathbb{R}$ in the central special case of positive trace, and in some other cases; where it applies, the construction provides meaningful bounds on the size of the realizing matrix in terms of the spectral gap.

The nonzero spectrum of a matrix is a "stable" or "eventual" invariant of a matrix. For a matrix over a field, an obvious finer invariant is the isomorphism class of the non-nilpotent part of its action as a linear transformation. The classification of matrices over a field by this invariant is the same as the classification up to shift equivalence over the field; for matrices over general rings, from the module viewpoint (see Sec.2), shift equivalence is the natural generalization of the isomorphism class of this nonnilpotent linear transformation. For some rings, an even finer invariant is the strong shift equivalence class. The Generalized Spectral Conjecture of Boyle and Handelman (in both forms below) heuristically is saying that only the obvious necessary spectral conditions constrain the eventual algebra of a primitive matrix over a subring of $\mathbb{R}$, regardless of the subring under consideration.

Generalized Spectral Conjecture (weak form, 1991) 1.3. *Suppose $\mathcal{R}$ is a subring of R and A is a square matrix over $\mathcal{R}$ whose nonzero spectrum satisfies the three necessary conditions of the Spectral Conjecture. Then A is SE over $\mathcal{R}$ to a primitive matrix.*

Generalized Spectral Conjecture (strong form, 1993) 1.4. *Suppose $\mathcal{R}$ is a subring of R and A is a square matrix over $\mathcal{R}$ whose nonzero spectrum satisfies the three necessary conditions of the Spectral Conjecture. Then A is SSE over $\mathcal{R}$ to a primitive matrix.*

The weak form was stated in [**?**, p.253] and [**?**, p.124]. The strong form was stated in [**?**, Sec. 8.4]), along with an explicit admission that the authors of the conjecture did not know if the conjectures were equivalent (not knowing if shift equivalence over a ring implies strong shift equivalence over it). Following [**?**] (see Theorem 2.1), we know now that the strong form of the Generalized Spectral Conjecture was not a vacuous generalization: there are subrings of $\mathbb{R}$ over which SE does not imply SSE (Example 3.5). The results of [**?**] also provide enough structure that we can prove Theorem 1.1, which shows that the two forms of the Generalized Spectral Conjecture are equivalent.

Note! In contrast to the statement of the Generalized Spectral Conjecture for *primitive* matrices, it is *not* the case that the existence of a strong shift equivalence over

$\mathcal{R}$ from a matrix A over $\mathcal{R}$ to a nonnegative matrix can in general be characterized by a spectral condition on A. There are dense subrings of $\mathbb{R}$ over which there are nilpotent matrices which are not SSE to nonnegative matrices (Remark 3.6).

There is some motivation from symbolic dynamics for pursuing the zero trace case of the GSC. The Kim-Roush and Wagoner primitive matrix counterexamples [**?**, **?**] to Williams' conjecture SE-$\mathbb{Z}_+ \implies$ SSE-$\mathbb{Z}_+$ rely absolutely on certain zero-positive patterns of traces of powers of the given matrix. We still do not know whether the refinement of SE-$\mathbb{Z}_+$ by SSE-$\mathbb{Z}_+$ is algorithmically undecidable or (at another extreme) if it allows some finite description involving such sign patterns. We are looking for any related insight.

2. Shift equivalence and strong shift equivalence

Suppose $\mathcal{R}$ is a subset of a semiring and $\mathcal{R}$ contains $\{0,1\}$. (For example, $\mathcal{R}$ could be $\mathbb{Z}, \mathbb{Z}_+, \{0,1\}, \mathbb{R}, \mathbb{R}_+, \dots$) Square matrices A, B over $\mathcal{R}$ (not necessarily of the same size) are *elementary strong shift equivalent over* $\mathcal{R}$ (ESSE-$\mathcal{R}$) if there exist matrices U, V over $\mathcal{R}$ such that $A = UV$ and $B = VU$. Matrices A, B are *strong shift equivalent over* $\mathcal{R}$ (SSE-$\mathcal{R}$) if there are a positive integer ℓ (the *lag* of the given SSE) and matrices $A = A_0, A_1, \dots, A_\ell = B$ such that A_{i-1} and A_i are ESSE-$\mathcal{R}$, for $1 \le i \le \ell$. For matrices over a subring of $\mathbb{R}$, the relation ESSE-$\mathcal{R}$ is never transitive. For example, if matrices A, B are ESSE over $\mathbb{R}$, $j > 1$ and $A^j \neq 0$, then $B^{j-1} \neq 0$; but if A is the $n \times n$ matrix such that $A(i, i+1) = 0$ for $1 \le i < n$ and $A = 0$ otherwise, then A is SSE-$\mathcal{R}$ to (0). Over any ring $\mathcal{R}$, the relation SSE-$\mathcal{R}$ on square matrices is generated by similarity over $\mathcal{R}$ ($U^{-1}AU \sim A$) and nilpotent extensions, $\left(\begin{smallmatrix} A & X \\ 0 & 0 \end{smallmatrix}\right) \sim A \sim \left(\begin{smallmatrix} 0 & X \\ 0 & A \end{smallmatrix}\right)$ [**?**].

Square matrices A, B over $\mathcal{R}$ are *shift equivalent over* $\mathcal{R}$ (SE-$\mathcal{R}$) if there exist a positive integer ℓ and matrices U, V over $\mathcal{R}$ such that the following hold:

$$\begin{aligned} A^\ell &= UV \qquad & B^\ell &= VU \\ AU &= UB \qquad & BV &= VA\ . \end{aligned}$$

Herem ℓ is the *lag* of the given SE. It is always the case that SSE over $\mathcal{R}$ implies SE over $\mathcal{R}$: from a given lag ℓ SSE one easily creates a lag ℓ SE [**?**]. For certain semirings $\mathcal{R}$, including above all $\mathcal{R} = \mathbb{Z}_+$, the relations of SSE and SE over $\mathcal{R}$ are significant for symbolic dynamics. The relations were introduced by Williams for the cases $\mathcal{R} = \mathbb{Z}_+$ and $\mathcal{R} = \{0,1\}$ to study the classification of shifts of finite type. Matrices over $\mathbb{Z}_+$ are SSE over $\mathbb{Z}_+$ if and only if they define topologically conjugate shifts of finite type. However, the relation SSE-$\mathbb{Z}_+$ to this day remains mysterious and is not even know to be decidable. In contrast, SE-$\mathbb{Z}_+$ is a tractable, decidable, useful and very strong invariant of SSE-$\mathbb{Z}_+$.

Suppose now $\mathcal{R}$ is a ring, and A is $n\times n$ over $\mathcal{R}$. T o see the shift equivalence relation SE-$\mathcal{R}$ more conceptually, recall that the direct limit G_A of $\mathcal{R}^n$ under the $\mathcal{R}$-module homomorphism $x\mapsto Ax$ is the set of equivalence classes $[x,k]$ for $x\in\mathcal{R}^n, k\in\mathbb{Z}_+$ under the equivalence relation $[x,k]\sim[y,j]$ if there exists $\ell>0$ such that $A^{j+\ell}x=A^{k+\ell}y$. G_A has a well defined group structure ($[x,k]+[y,j]=[A^kx+A^jy,j+k]$) and is an $\mathcal{R}$-module ($r:[x,k]\mapsto[xr,k]$). A induces an $\mathcal{R}$-module isomorphism $\hat{A}:[x,k]\mapsto[Ax,k]$ with inverse $[x,k]\mapsto[x,k+1]$. G_A becomes an $\mathcal{R}[t]$ module (also an $\mathcal{R}[t,t^{-1}]$ module) with $t:[x,k]\mapsto[x,k+1]$. A and B are SE-$\mathcal{R}$ if and only if these $\mathcal{R}[t]$-modules are isomorphic (equivalently, if and only if they are isomorphic as $\mathcal{R}[t,t^{-1}]$ modules). If the square matrix A is $n\times n$, then $I-tA$ defines a homomorphism $\mathcal{R}^n\to\mathcal{R}^n$ by the usual multiplication $v\mapsto(I-tA)v$, and $\operatorname{cok}(I-tA)$ is an $\mathcal{R}[t]$-module which is isomorphic to the $\mathcal{R}[t]$-module G_A. For more detail and references on these relations (by no means original to us) see [**?**, **?**].

Williams introduced SE and SSE in the 1973 paper [**?**]. For any principal ideal domain $\mathcal{R}$, Effros showed SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ in the 1981 monograph [**?**] (see [**?**] for Williams' proof for the case $\mathcal{R}=\mathbb{Z}$). In the 1993 paper [**?**], Boyle and Handelman extended this result to the case that $\mathcal{R}$ is a Dedekind domain (or, a little more generally, a Prüfer domain). Otherwise, the relationship of SE and SSE of matrices over a ring remained open until the recent paper [**?**], which explains the relationship in general as follows.

Theorem 2.1. [**?**] *Suppose A,B are SE over a ring $\mathcal{R}$.*

(1) There is a nilpotent matrix N over $\mathcal{R}$ such that B is SSE over $\mathcal{R}$ to the matrix $A\oplus N=\begin{pmatrix}A&0\\0&N\end{pmatrix}$.

(2) The map $[I-tN]\to[A\oplus N]_{SSE}$ induces a bijection from $\mathrm{NK}_1(\mathcal{R})$ to the set of SSE classes of matrices over $\mathcal{R}$ which are in the SE-$\mathcal{R}$ class of A.

We will say just a little now about $\mathrm{NK}_1(\mathcal{R})$, a group of great importance in algebraic K-theory; for more background, we have found [**?**, **?**, **?**] very helpful. $\mathrm{NK}_1(\mathcal{R})$ is the kernel of the map $K_1(\mathcal{R}[t])\to K_1(\mathcal{R})$ induced by the ring homomorphism $\mathcal{R}[t]\to\mathcal{R}$ which sends t to 0. The finite matrix $I-tN$ corresponds to the matrix $I-(tN)_\infty$ in the group $\mathrm{GL}(\mathcal{R}[t])$ (with I denoting the $\mathbb{N}\times\mathbb{N}$ identity matrix and $(tN)_\infty$ the $\mathbb{N}\times\mathbb{N}$ matrix which agrees with tN in an upper left corner and is otherwise zero). Every class of $\mathrm{NK}_1(\mathcal{R})$ contains a matrix of the form $I-(tN)_\infty$ with N nilpotent over $\mathcal{R}$. $\mathrm{NK}_1(\mathcal{R})$ is trivial for many rings (e.g., any field, or more generally any left regular Noetherian ring) but not for all rings. If $\mathrm{NK}_1(\mathcal{R})$ is not trivial, then it is not finitely generated as a group. From the established theory, it is easy to give an example of a subring $\mathcal{R}$ of $\mathbb{R}$ for which $\mathrm{NK}_1(\mathcal{R})$ is not trivial (Example 3.5).

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Given a square matrix M over $\mathbb{R}$, let λ_M denote its spectral radius and define the matrix $|M|$ by $|M|(i,j) = |M(i,j)|$.

By Theorem 2.1, let N be a nilpotent matrix such that B is SSE over $\mathcal{R}$ to the matrix $\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$. Suppose M is a matrix SSE over $\mathcal{R}$ to N and M also satisfies the following conditions:

(1) $\lambda_{|3M|} < \lambda_A$
(2) For all positive integers n, $\mathrm{trace}(|3M|^n) \leq \mathrm{trace}(A^n)$.
(3) For all positive integers n and k, if $\mathrm{tr}(|3M|^n) < \mathrm{tr}(A^n)$, then $\mathrm{tr}(|3M|^{nk}) < \mathrm{tr}(A^{nk})$.

Then by the Submatrix Theorem (Theorem 3.1 of [**?**]), there is a primitive matrix C SSE over $\mathcal{R}$ to A such that $|3M|$ is a proper principal submatrix of C. Without loss of generality, let this submatrix occupy the upper left corner of C. Define M_0 to be the matrix of size matching C which is M in its upper left corner and which is zero in other entries. Then B is SSE over $\mathcal{R}$ to the matrix $\begin{pmatrix} C & 0 \\ 0 & M_0 \end{pmatrix}$. Choose $\epsilon \in \mathcal{R}$ such that $1/3 < \epsilon < 2/3$ and compute

$$\begin{pmatrix} I & -\epsilon I \\ 0 & I \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} I & \epsilon I \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & \epsilon(C - M_0) \\ 0 & M_0 \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} C & \epsilon(C - M_0) \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} = \begin{pmatrix} (1-\epsilon)C + \epsilon M_0 & \epsilon(C - M_0) \\ (1-\epsilon)(C - M_0) & \epsilon C + (1-\epsilon)M_0 \end{pmatrix} := G \; .$$

The matrix G is SSE over $\mathcal{R}$ to B, and it is nonnegative. The diagonal blocks have positive entries wherever C does; because C is primitive, there is a $j > 0$ such that $C^j > 0$, and therefore the diagonal blocks of G^j are also positive. Because neither offdiagonal block of G is the zero block, it follows that G is primitive.

So, it suffices to find M SSE over $\mathcal{R}$ to N satisfying the conditions (1)-(3) above. Choose K such that $\mathrm{tr}(A^k) > 0$ for all $k > K$. Let n be the integer such that N is $n \times n$, and let J by the integer provided by Proposition 3.4 given n and K. Given this J, choose $\epsilon > 0$ such that for any $J \times J$ matrix M with $||M||_\infty < \epsilon$, we have $\lambda_{3|M|} < \lambda_A$ and for $k > K$ we also have $\mathrm{tr}(|3M|^k) < \mathrm{tr}(A^k)$. Now let $\delta > 0$ be as provided by Proposition 3.4 for this ϵ.

If we can now find an $n \times n$ nilpotent matrix N' which is SSE over $\mathcal{R}$ to N and satisfies $||N'|| < \delta$, then we can apply Proposition 3.4 to this N' to produce a matrix M SSE over $\mathcal{R}$ to N and with $||M|| < \epsilon$ and with $\mathrm{tr}(M^k) = 0$ for $1 \leq k \leq K$. This matrix M will satisfy the conditions (1)-(3).

Pick $\gamma > 0$ such that $||\gamma N||_\infty < \delta$. There is a matrix U in $\mathrm{SL}(n, \mathbb{R})$ such that $U^{-1}NU = \gamma N$. The matrix U is a product of basic elementary matrices over $\mathbb{R}$, and these can be approximated arbitrarily closely by basic elementary matrices over $\mathcal{R}$. Consequently there is a matrix V in $\mathrm{SL}(n, \mathcal{R})$ such that $||V^{-1}NV||_\infty < \delta$. Choose $N' = V^{-1}NV$. □

To prove the Proposition 3.4 on which the proof of Theorem 1.1 depends, we use a correspondence proved in [**?**]. We need some definitions.

Given a finite matrix A, let A_∞ denote the $\mathbb{N}\times\mathbb{N}$ matrix which has A as its upper left corner and is otherwise zero. In any $\mathbb{N}\times\mathbb{N}$ matrix, I denotes the infinite identity matrix. Given a ring R, $\mathrm{El}(R)$ is the group of $\mathbb{N} \times \mathbb{N}$ matrices over $R[t]$, equal to the infinite identity matrix except in finitely many entries, which are products of basic elementary matrices (these basic matrices are by definition equal to I except perhaps in a single offdiagonal entry). For finite matrices A, B, the matrices $I-A_\infty$ and $I-B_\infty$ are $\mathrm{El}(R[t])$ equivalent if there are matrices U, V in $\mathrm{El}(R[t])$ such that $U(I - A_\infty)V = I - B_\infty$.

Definition 3.1. *Given a finite matrix A over $t\mathcal{R}[t]$, choose $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $A_1, \ldots A_k$ are $n \times n$ matrices over $\mathcal{R}$ such that*

$$A_\infty = \sum_{i=1}^{k} t^i (A_i)_\infty$$

and define a finite matrix $\mathcal{A}^\sharp = \mathcal{A}^{\sharp(k,n)}$ over $\mathcal{R}$ by the following block form, in which every block is $n \times n$:

$$\mathcal{A}^\sharp = \begin{pmatrix} A_1 & A_2 & A_3 & \ldots & A_{k-2} & A_{k-1} & A_k \\ I & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & I & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & I & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & I & 0 \end{pmatrix} .$$

In the definition, there is some freedom in the choice of $\mathcal{A}^\sharp$: k can be increased by using zero matrices, and n can be increased by filling additional entries of the A_i with zero. These choices do not affect the SSE-$\mathcal{R}$ class of $\mathcal{A}^\sharp$.

Theorem 3.2. [?] *Let $\mathcal{R}$ be a ring. Then there is a bijection between the following sets:*

- *the set of $\mathrm{El}(\mathcal{R}[t])$ equivalence classes of $\mathbb{N} \times \mathbb{N}$ matrices $I - A_\infty$ such that A is a finite matrix over $t\mathcal{R}[t]$*
- *the set of SSE-$\mathcal{R}$ classes of square matrices over $\mathcal{R}$.*

The bijection from $\mathrm{El}(\mathcal{R}[t])$ *equivalence classes to SSE-*$\mathcal{R}$ *classes is induced by the map* $I - A_\infty \mapsto \mathcal{A}^\sharp$*. The inverse map (from the set of SSE-*$\mathcal{R}$ *classes) is induced by the map sending* A *over* $\mathcal{R}$ *to the* $\mathbb{N}\times\mathbb{N}$ *matrix* $I - tA$*.*

By the degree of a matrix with polynomial entries we mean the maximum degree of its entries. If M is a matrix over $\mathbb{R}[t]$, with entries $M(i,j) = \sum_{i,j,k} m_{ijk}t^k$, then we define $||M|| = \max_{k>0}\max_{i,j}|m_{ijk}|$. If M is a matrix over $\mathbb{R}$, with $M(i,j) = m_{ij}$, then $||M||_\infty$ is the usual sup norm, $||M||_\infty = \max_{i,j}|m_{ij}|$.

Lemma 3.3. *Suppose* $\mathcal{R}$ *is a dense subring of* $\mathbb{R}$ *and* A *is an* $n\times n$ *matrix of degree* d *over* $t^k\mathcal{R}[t]$*, with entries* $a_{ij} = \sum_{1\le r\le d} a_{ij}^{(r)}t^r$*. Suppose* $\sum_{i=1}^n a_{ii}^{(k)} = 0$ *and* $||A|| \le \frac{1}{4n^2}$*. Then there is an* $n\times n$ *matrix* B *over* $t^{k+1}\mathcal{R}[t]$ *such that* $I - A_\infty$ *is* $\mathrm{El}(\mathcal{R}[t])$ *equivalent to* $I - B_\infty$ *and the following hold:*

(1) $\mathrm{degree}(B) \ \le\ \mathrm{degree}(A) + 3k$*.*
(2) $||B|| \ \le\ 4n^3||A||$ *.*

Proof. For finite square matrices $I-C$ and $I-D$, we use $I-C\sim I-D$ to denote elementary equivalence over $\mathcal{R}[t]$ of $I-C_\infty$ and $I-D_\infty$. We have

$$
\begin{aligned}
I - A \ &= \begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix} \\
&\sim \begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} & a_{11}^{(k)}t^k \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} & a_{22}^{(k)}t^k \\ \vdots & & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} & a_{nn}^{(k)}t^k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} := I - A_1 \ .
\end{aligned}
$$

In order, apply the following elementary operations:

(1) For $1\le j\le n$, add column $n+1$ to column j of $I-A_1$, to produce a matrix $I-A_2$. Then $\mathrm{degree}(A_2) = \mathrm{degree}(A)$; the diagonal entries of A_2 lie in $t^{k+1}\mathcal{R}[t]$; and $||A_2|| \le 2||A_1|| = 2||A||$. Every entry in row $n+1$ of $I-A_2$ equals 1. (By definition these entries have no impact on $||A_2||$.)
(2) For $1\le i\le n$, add (-1)(row i) of $(I-A_2)$ to row $n+1$ to form $I-A_3$. Then the entries of A_3 lie in $t^k\mathcal{R}[t]$, and the diagonal entries of A_3 still lie in $t^{k+1}\mathcal{R}[t]$, since $\sum_{i=1}^n a_{ii}^{(k)} = 0$. We have $||A_3|| \le n||A_2|| \le 2n||A|| < 1$ and $\mathrm{degree}(A_3) \le \mathrm{degree}(A)$.

(3) For $1 \leq i \leq n$, add $(-a_{ii}^{(k)} t^k)$(row $n+1$) of $(I - A_3)$ to row i to form $I - A_4$. In block form,

$$I - A_4 \; = \; \begin{pmatrix} I - A_5 & 0 \\ x & 1 \end{pmatrix}$$

in which A_5 is $n \times n$ and $x = (x_1 \cdots x_n)$. Adding multiples of column $n+1$ to columns $1, \dots, n$ to clear out x, we see $I - A_5 \sim I - A$. We have degree$(A_5) \leq$ degree$(A) + k$ and

$$\begin{aligned} ||A_5|| \; &\leq \; ||A_3|| + (||A||)(||A_3||) \\ &\leq \; 2||A_3|| \; \leq \; 4n||A|| \; < \; 1 \; . \end{aligned}$$

In A_5, the diagonal terms lie in $t^{k+1}\mathcal{R}[t]$ and the offdiagonal terms lie in $t^k\mathcal{R}[t]$. In the next two steps, we apply elementary operations to clear the degree k terms outside the diagonal. We use part of a clearing algorithm from [**?**].

(4) Let b_{ij} be the coefficient of t^k in $A_5(i,j)$. For $2 \leq i \leq n$, add $(-b_{1j}t^k)$(row j) to row 1. Continuing in order for rows $i = 2, \dots, n-1$: for $i+1 \leq j \leq n$, add $(-b_{ij}t^k)$(row j) to row i. Let $(I - A_6)$ be the resulting matrix. The entries of A_6 on and above the diagonal lie in $t^{k+1}\mathcal{R}[t]$. We have

$$\text{degree}(A_6) \; \leq \; \text{degree}(A_5) + k \; \leq \; \text{degree}(A) + 2k$$

and

$$\begin{aligned} ||A_6|| \; &\leq \; ||A_5|| + (n-1)||A_5||^2 \\ &\leq \; n||A_5|| \; \leq \; 4n^2||A|| \leq 1 \; . \end{aligned}$$

(5) Let c_{ij} denote the coefficient of t^k in $A_6(i,j)$. For $2 \leq j \leq n$, add $(-c_{j1}t^k)$(column j) of A_6 to column 1. Continuing in order for columns $i = 2, \dots, n-1$: for $i+1 \leq j \leq n$, add $(-c_{ji})$(column j) to column i. For the resulting matrix $(I - B)$, the entries of B lie in $t^{k+1}\mathcal{R}[t]$, with

$$\text{degree}(B) \; \leq \; \text{degree}(A_6) + k \; \leq \; \text{degree}(A) + 3k$$

and

$$\begin{aligned} ||B|| \; &\leq \; ||A_6|| + (n-1)||A_6||^2 \\ &\leq \; n||A_6|| \; \leq \; 4n^3||A|| \; . \end{aligned}$$

□

Proposition 3.4. *Suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$, $n \in \mathbb{N}$ and $K \in \mathbb{N}$. Then there is a J in $\mathbb{N}$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that the following holds: if N is a nilpotent $n \times n$ matrix over $\mathcal{R}$ and $||N||_\infty < \delta$, then there is a $J \times J$ matrix M over $\mathcal{R}$ such that*

(1) M *is SSE over* $\mathcal{R}$ *to* N*,*
(2) $\mathrm{tr}\,(|M|^k) = 0$ *for* $1 \le k \le K$*, and*
(3) $||M||_\infty < \epsilon$ *.*

Proof. Because N is nilpotent, $\mathrm{tr}(N^k) = 0$ for all positive integers k. Set $B_0 = tN$. We define matrices $B_1, \dots, B_K$ recursively, letting $I - B_{k+1}$ be the matrix $I - B$ provided by Lemma 3.3 from input $I - A = I - B_k$. The conditions of the lemma are satisfied recursively, because the (zero) trace of the kth power of the nilpotent matrix $(B_k)^\sharp$ must be (in the terminology of the lemma) $\sum_i a_{ii}^{(k)}$. The matrix B_K is $n \times n$ with entries of degree at most

$$d := 1 + 3(1) + 3(2) + \cdots + 3(K) = 1 + 3K(K+1)/2 \ .$$

Let $(B_K)_i$ be the matrices, $1 \le i \le d$, such that $B_K = \sum_{i=1}^d (B_K)_i t^i$. Define M to be the matrix $(B_K)^\sharp$, an $nd \times nd$ matrix over $\mathcal{R}$ which is SSE over $\mathcal{R}$ to N. Set $J = nd$.

It is now clear from condition (2) of Lemma 3.3 and induction that given $\epsilon > 0$, there is a $\delta > 0$ such that $||N|| < \delta$ implies $||(B_K)|| < \epsilon$. (We are not trying to optimize estimates.) With $K > 1$ (without loss of generality), we have $||B_K|| = ||(B_K)^\sharp||_\infty$. This finishes the proof. □

Example 3.5. There are subrings of $\mathbb{R}$ with nontrivial NK_1. For example, let $\mathcal{R} = \mathbb{Q}[t^2, t^3, z, z^{-1}]$. By the Bass-Heller-Swan Theorem (see [?], 3.2.22) for any ring $\mathcal{S}$, there is a splitting $K_1(\mathcal{S}[z, z^{-1}]) \cong K_1(\mathcal{S}) \oplus K_0(\mathcal{S}) \oplus \mathrm{NK}_1(\mathcal{S}) \oplus \mathrm{NK}_1(\mathcal{S})$, which implies $\mathrm{NK}_1(\mathcal{S}[z, z^{-1}])$ always contains a copy of $\mathrm{NK}_0(\mathcal{S})$. An elementary argument (see for example exercise 3.2.24 in [?]) shows that $\mathrm{NK}_0(\mathbb{Q}[t^2, t^3]) \neq 0$, so $\mathrm{NK}_1(\mathbb{Q}[t^2, t^3, z, z^{-1}])$ is non-zero. Since $\mathbb{Q}[t^2, t^3, z, z^{-1}]$ can be realized as a subring of $\mathbb{R}$ (by an embedding sending t, z to algebraically independent transcendentals in $\mathbb{R}$) this provides an example of a subring $\mathcal{R}$ of $\mathbb{R}$ for which $\mathrm{NK}_1(\mathcal{R})$ is not zero, and therefore shift equivalence over $\mathcal{R}$ does not imply strong shift equivalence over $\mathcal{R}$.

It is possible to produce explicit examples by tracking through the exact sequences behind the argument of the last paragraph. This is done in [?], and for $\mathcal{R} = \mathbb{Q}[t^2, t^3, z, z^{-1}]$ yields the following matrix over $\mathcal{R}[s]$,

$$I - M = \begin{pmatrix} 1 - (1 - z^{-1})s^4t^4 & (z-1)(s^2t^2 - s^3t^3) \\ (1 - z^{-1})(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & 1 + (z-1)(s^4t^4) \end{pmatrix} \ ,$$

which is nontrivial as an element of $\mathrm{NK}_1(\mathcal{R})$. Writing M as

$$M = \begin{pmatrix} (1 - z^{-1})s^4t^4 & (1-z)(s^2t^2 - s^3t^3) \\ (z^{-1} - 1)(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & (1-z)(s^4t^4) \end{pmatrix} = \sum_{i=1}^{5} s^i M_i$$

with the M_i over $\mathcal{R}$, we obtain (see [?]) a nilpotent matrix N over $\mathcal{R}$,

$$N = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & (1-z)t^2 & 0 & (1-z)(-t^3) & (1-z^{-1})t^4 & 0 & 0 & 0 \\ 0 & 0 & (z^{-1}-1)t^2 & 0 & (z^{-1}-1)t^3 & 0 & (z^{-1}-1)t^4 & (1-z)t^4 & (z^{-1}-1)t^5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which is nontrivial as an element of $\mathrm{Nil}_0(\mathcal{R})$, as is the matrix N' obtained by removing the last row and the last column from N.

The matrix N' is 9×9. We don't have a smaller example, and we don't have a decent example of two positive matrices which are shift equivalent but not strong shift equivalent over a subring of $\mathbb{R}$.

Remark 3.6. Suppose $\mathcal{R}$ is a subring of $\mathbb{R}$ and N is a nonnegative nilpotent matrix over $\mathcal{R}$. Then there is a permutation matrix P such that $P^{-1}NP$ is triangular with zero diagonal. Using elementary SSEs of the block form

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} X \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

we see that $P^{-1}NP$ (and hence N) is SSE over $\mathcal{R}$ to $[0]$. By Theorem 2.1, with $A = 0$, it follows that a nilpotent matrix N is SSE over $\mathcal{R}$ to a nonnegative matrix if and only if $[I - tN_\infty]$ is trivial in $\mathrm{NK}_1(\mathcal{R})$. Therefore, if (and only if) $\mathrm{NK}_1(\mathcal{R})$ is nontrivial, there will be nilpotent matrices over $\mathcal{R}$ which cannot be SSE over $\mathcal{R}$ to a nonnegative matrix. The matrix N in Example 3.5 is one such example.

4. Reflections on the Generalized Spectral Conjecture

Is the Generalized Spectral Conjecture true?

For $\mathcal{R} = \mathbb{Z}$, the Spectral Conjecture is true [**?**]. The GSC is true for $\mathcal{R} = \mathbb{Z}$ for a given Δ if every entry of Δ is a rational integer [**?**]. There is not much more direct evidence for the GSC for $\mathcal{R} = \mathbb{Z}$, but we know of no results which cast doubt.

From here, suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$. As noted earlier, the Spectral Conjecture is almost surely true. Theorem 1.1 removes the possibility that the very subtle algebraic invariants following from Theorem 2.1 could be an obstruction to the GSC. The GSC was proved in [**?**] in the following cases:

(1) when the nonzero spectrum is contained in $\mathcal{R}$, and $\mathcal{R}$ is a Dedekind domain with a nontrivial unit;
(2) when the nonzero spectrum has positive trace and either (i) the spectrum is real or (ii) the minimal and characteristic polynomials of the given matrix are equal up to a power of the indeterminate.

The following Proposition (almost explicit in [**?**, Appendix 4]) is more evidence for the GSC in the positive trace case.

Proposition 4.1. *Suppose the Generalized Spectral Conjecture holds for matrices of positive trace for the ring* $\mathbb{R}$. *Then it holds for matrices of positive trace for every dense subring* $\mathcal{R}$ *of* $\mathbb{R}$.

Proof. Let A be a square matrix over $\mathcal{R}$ of positive trace which over $\mathbb{R}$ is SSE to a primitive real matrix B. We need to show that A is SSE over $\mathcal{R}$ to a primitive matrix.

By [**?**] (or the alternate exposition [**?**, Appendix B]), because B is primitive with positive trace, there is a positive matrix B_1 SSE over $\mathbb{R}$ (in fact over $\mathbb{R}_+$) to B. And then, by arguments in [**?**], for some m there are $m \times m$ matrices A_2, B_2 (obtained through row splittings of A and B_1), with B_2 positive, such that A is SSE over $\mathcal{R}$ to A_2; B_1 is SSE over $\mathbb{R}$ (in fact over $\mathbb{R}_+$) to a positive matrix B_2; and there is a matrix U in $\mathrm{SL}(m, \mathbb{R})$ such that $U^{-1}A_2U = B_2$. Because $\mathrm{SL}(m, \mathcal{R})$ is dense in $\mathrm{SL}(m, \mathbb{R})$, and B_2 is positive, there is a V in $\mathrm{SL}(m, \mathcal{R})$ such that $V^{-1}A_2V$ is positive. This matrix $(V^{-1}A_2)(V)$ is SSE over $\mathcal{R}$ to the matrix $(V)(V^{-1}A_2) = A$. $\square$

After more than 20 years, the GSC remains open even in the case $\mathcal{R} = \mathbb{R}$. Still, the GSC seems correct. What we lack is a proof.