Spring 2012 - Math 437 Section 0101

Homework #1 - **Due Tuesday Feb. 7**

1. Classify the following sets as open, closed or neither. For each set, give its interior and its boundary.

   (a) $I_1 = [1, 3)$, $I_2 = (-\infty, 2] \cup \{4\}$, $I_3 = \{x \in (0, 1)\,;\, x \text{ is rational}\}$

   (b) $A_1 = [0, 1] \times [0, 1]$, $A_2 = \{x \in \mathbb{R}^2\,;\, \|x\| = 1\}$

2. Let f be defined by $f(x, y) = (e^x, x - y)$. Find $A \subset \mathbb{R}^2$ such that $f : \mathbb{R}^2 \to A$ is an homeomorphism. Find f^{-1}.

3. For each of the following linear transformations, state whether L is injective, surjective or bijective (justify your answer). Compute L^{-1} when it exists.

   (a) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L(x_1, x_2) = (2x_1 - x_2, x_1 + 3x_2)$.

   (b) $L : \mathbb{R}^2 \to \mathbb{R}^3$ given by $L(x_1, x_2) = (x_1 + x_2, x_2, 2x_1 - x_2)$.

   (c) $L : \mathbb{R}^3 \to \mathbb{R}^3$ linear transformation with matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 2 & -2 & 2 \end{bmatrix}$$

4. Compute all first order partial derivatives of the following functions

   (a) $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2y^5 + xz^2$

   (b) $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = e^{x^2}(\cos y + \cos z)$

   (c) $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (e^{2x}, \cos(x + y))$

5. Find the derivative df of the following functions

   (a) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy^2$

   (b) $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (\sin(xy), \sin(x)\sin(y), e^{xy})$

   (c) $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x - 5y, 2x + 3y, 4x - y)$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + y, 4xy)$. Compute the derivative of f. For which (x, y) is $df_{(x,y)}$ bijective?

7. (a) If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f'(x) \neq 0$ for all $x \in \mathbb{R}$, show that f is injective in $\mathbb{R}$.

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $df_{(x,y)}$ is invertible for all $(x, y) \in \mathbb{R}^2$, but that f is not injective in $\mathbb{R}^2$.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$f(tx) = tf(x) \quad \text{for all } x \in \mathbb{R}^n,\ x \neq 0, \text{ and for all } t > 0$$

   Show that f is a linear transformation.

   Hint: Differentiate the equality $f(tx) = tf(x)$ with respect to t.