Spring 2012 - Math 437 Section 0101

Homework #10 - **Due May 1st**

1. Consider the following forms in $\Omega^1(\mathbb{S}^1)$. For each determine whether it is exact or not, and if not, find $\lambda \in \mathbb{R}$ such that $\omega - \lambda d\theta$ is exact.

   (a) $\omega = (x+y)\,dx + (x-y)\,dy$

   (b) $\omega = -(x+y)\,dx + (x-y)\,dy$

   (c) $\omega = y^3 dx - x^3 dy$

2. We recall that $H^1(\mathbb{S}^3) = H^2(\mathbb{S}^3) = 0$. Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a smooth function.

   (a) For a given $\beta \in \Omega^2(\mathbb{S}^2)$, we define $\alpha = f^*\beta$. Show that there exists $\eta \in \Omega^1(\mathbb{S}^3)$ such that $d\eta = \alpha$ and show that the integral

   $$\int_{\mathbb{S}^3} \alpha \wedge \eta$$

   does not depend on a particular choice of η such that $d\eta = \alpha$.
   Hint: Let η_1 and η_2 be such that $d\eta_i = \alpha$, and compute $\int_{\mathbb{S}^3} d(\eta_1 \wedge \eta_2)$.

   (b) We consider all $\beta \in \Omega^2(\mathbb{S}^2)$ such that $\int_{\mathbb{S}^2} \beta = 1$. For any such β, we define $\alpha = f^*\beta$ and let η be such that $d\eta = \alpha$. Show that the integral

   $$\int_{\mathbb{S}^3} \alpha \wedge \eta$$

   does not depend on a particular choice of β (so it only depends on f).
   Hint: Let β_1 and β_2 be two such forms. Show that the corresponding α_i and η_i can be chosen so that

   $$\alpha_2 = \alpha_1 + f^*(d\gamma), \quad \eta_2 = \eta_1 + f^*\gamma$$

   for some $\gamma \in \Omega^1(\mathbb{S}^2)$ (and remember that $\Omega^3(\mathbb{S}^2) = \{0\}$).

3. Given a k-form in $\mathbb{R}^n$, we define an $(n-k)$-form $*\omega$ by setting

   $$*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma (dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}})$$

   and extending it linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$ and $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, 2, \ldots, n)$ and σ is 0 or 1 according to the permutation is even or odd.

(a) Check that if

$$\omega = a_1 dx_1 + a_2 dx_2 \in \Omega^1(\mathbb{R}^2)$$

then

$$*\omega = a_1 dx_2 - a_2 dx_1.$$

(b) Check that if

$$\omega = a_1 dx_2 \wedge dx_3 + a_2 dx_3 \wedge dx_1 + a_3 dx_1 \wedge dx_2$$

then

$$*\omega = a_1 dx_1 + a_2 dx_2 + a_3 dx_3.$$

(c) With the ω of (b), compute $**\omega$.

(d) Let $F = (f_1, f_2, f_3)$ be a function $F : \mathbb{R}^3 \to \mathbb{R}^3$, and define $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = F \cdot dx$. Show that

$$d * \omega = \text{div } F dx_1 \wedge dx_2 \wedge dx_3$$

(we also write $\text{div } F = *d*\omega$).

(e) Similarly, show that $\text{curl } F \cdot dx = *d\omega$.

(f) Given a function $f : \mathbb{R}^3 \to \mathbb{R}$, we define the laplacian

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Show that

$$\Delta f = *d*df$$

4. Let U be a path-connected open subset of $\mathbb{R}^n$. A function $\gamma : [a,b] \to U$ is a piecewise differentiable curve if it is continuous and if there exists $t_0 = a < t_1 < \cdots < t_k = b$ such that the restriction of γ to $[t_j, t_{j+1}]$ (denoted γ_j) is differentiable for all j. Let $\omega = \sum a_i dx_i$ be a 1-form on U, we can then define

$$\int_\gamma \omega = \sum_j \int_{\gamma_j} \omega$$

for any piecewise differentiable curve γ in U.

(a) Prove that if ω is exact in U, then $\int_\gamma \omega$ only depends on the endpoints $\gamma(a)$ and $\gamma(b)$ for any piecewise differentiable curve γ in U.

(b) Prove that if $\int_\gamma \omega$ only depends on the endpoints $\gamma(a)$ and $\gamma(b)$ for any piecewise differentiable curve γ in U, then ω is exact.

Hint: Fix $p \in U$ and define $f(x) = \int_\gamma \omega$ where γ is a piecewise differentiable curve joining p to x. Prove that $\frac{\partial f}{\partial x_i} = a_i$ by considering curves $\gamma_i : t \mapsto x + te_i$.