1. (a) $I_1=[1,3)$ Neither open nor closed
$Int(I_1)=(1,3)$ $\partial I_1=\{1,3\}$

~~(b)~~ $I_2=(-\infty,2]\cup\{4\}$ closed
$Int(I_2)=(-\infty,2)$ $\partial I_2=\{2,4\}$

~~(c)~~ $I_3=\mathbb{Q}\cap(0,1)$ Neither
$Int(I_3)=\Phi$, $\partial I_3=[0,1]$

(b) $A_1=[0,1]\times[0,1]$ closed
$Int(A_1)=(0,1)\times(0,1)$ $\partial A_1=[0,1]\times[0,1]\setminus(0,1)\times(0,1)$
$=\{0\}\times[0,1]\cup\{1\}\times[0,1]\cup[0,1]\times\{0\}\cup[0,1]\times\{1\}$

$A_2=\{x\in\mathbb{R}^2 \mid \|x\|=1\}$ closed
$Int A_2=\phi$. $\partial A_2=A_2$

2. $f(x,y)=(e^x, x-y)$ $f^{-1}(x,y)=(\ln x, \ln x-y)$
$A=(0,+\infty)\times(-\infty,+\infty)$

3. (a) $L$ bijective $L^{-1}=\frac{1}{7}\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}$

(b) injective

(c) $\det A=0$. ~~not~~ neither injective nor surjective

4 (a) $f=x^2y^5+xz^2$ $f_x=2xy^5+z^2$, $f_y=5x^2y^4$, $f_z=2xz$

(b) $f=e^{x^2}(\cos y+\cos z)$ $f_x=2e^{x^2}x(\cos y+\cos z)$ $f_y=-e^{x^2}\sin y$
$f_z=-e^{x^2}\sin z$

(c) $f=(e^{2x}, \cos(x+y))$ $df=\begin{pmatrix}2e^{2x} & 0\\ -\sin(x+y) & -\sin(x+y)\end{pmatrix}$

5 (a) $f=xy^2 \Rightarrow df=y^2dx+2xy\,dy$

(b) $f(x,y) = (\sin(xy), \sin x \sin y, e^{xy})$

$$df = \begin{pmatrix} \cos(xy)\, y\, dx & \cos(xy)\, x\, dy \\ \cos x \sin y\, dx & \sin x \cos y\, dy \\ e^{xy} y\, dx & e^{xy} x\, dy \end{pmatrix}$$

(c) $f(x,y) = (x-5y, 2x+3y, 4x-y)$

$$df = \begin{pmatrix} dx - 5dy \\ 2dx + 3dy \\ 4dx - dy \end{pmatrix}$$

6 $f(x,y) = (x+y, 4xy)$

$$df = \begin{pmatrix} 1 & 1 \\ 4y & 4x \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \det \begin{pmatrix} 1 & 1 \\ 4y & 4x \end{pmatrix} = 4(x-y)$$

$df_{(x,y)}$ is bijective iff $x \neq y$

7 (a) suppose $f$ is not injective, then there exists 2 points $x, y \in \mathbb{R}$ $f(x) - f(y) = 0$. Intermediate value theorem tells us, there exists one point $\xi$ lies in between $x$ and $y$ such that $f'(\xi) = 0$. Contradiction

(b) $$df = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} \qquad \det(df) = e^{2x} \neq 0 \Rightarrow df \text{ is invertib}$$

But $f$ is not injective. For instance $f(0,0) = f(0, 2\pi)$

8. $f(tx) = t f(x)$. Take derivative w.r.t. $t$

$f'(tx) \cdot x = f(x)$, $f(0) = 0$

The fact $f$ is differentiable at $x=0$ means, there exist constant vector $A$

$\lim_{\|x\| \to 0} \frac{\|f(x) - f(0) - Ax\|}{\|x\|} = 0$. This implies all directional derivatives ~~$\lim_{t\to 0} \frac{f(tx) - f(0)}{t}$~~ exists and equal to $A$ (Gateaux derivative)

so we can pass to the limit $\lim_{t \to 0} f'(tx) \cdot x = Ax = f(x)$