Spring 2012 - Math 437 Section 0101

Homework #2 - **Due February 14**

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (\sin(x^2 + y^2), \cos(x^2 + y^2))$.
   (a) Compute the derivative of f.
   (b) Show that $df_{(x,y)}$ is nowhere bijective?
   (c) Determine $f(\mathbb{R}^2)$ and explain why this is consistent with part (b).

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $df_x(h) = 0$ for all $x \in \mathbb{R}^2$ and $h \in \mathbb{R}^2$. Show that f is constant in $\mathbb{R}^2$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f cannot be injective.

   Hint: If, for instance, $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$, then consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x, y) = (f(x, y), y)$.

4. Let $U = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $f : U \to U$ be defined by

$$f(x, y) = (x^2 - y^2, 2xy)$$

   Show that f is a local diffeomorphism at all point in U. Is f is diffeomorphism on U?

5. A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be bilinear if

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$$

$$f(cx, y) = cf(x, y), \quad f(x, cy) = cf(x, y)$$

   Prove that any bilinear function is differentiable, and compute its derivative.

6. Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 \,;\, z = x^2 - y^2\}$ is a regular surface. Show that $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(u, v) = (u + v, u - v, 4uv), \quad (u, v) \in \mathbb{R}^2$$

   is a parametrization for S.

7. Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3 \,;\, x^2 + y^2 = 1\}$ is a regular surface. Find parametrizations covering the whole set.

8. Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value of f and yet $f^{-1}(0)$ is a regular surface.