1 (a) $Df = \begin{pmatrix} 2x\cos(x^2+y^2) & -2x\sin(x^2+y^2) \\ 2y\cos(x^2+y^2) & -2y\sin(x^2+y^2) \end{pmatrix}$

(b) $\det Df = 0$ nowhere bijective

(c) $\sin^2(x^2+y^2) + \cos^2(x^2+y^2) = 1$

Since $(x,y) \in \mathbb{R}^2$, then x^2+y^2 covers $[0, 2\pi)$

This implies $f(\mathbb{R}^2) = S^1$. locally f maps 2-dim open set to a curve

This is consistent with (b)

2. Suppose f is not a constant, then $\exists\, x, y$ such that $f(x) \neq f(y)$, connect x and y using a straight line, we get another function

$g(t) = f(tx + (1-t)y)$, such that $g(0) = f(y)$, $g(1) = f(x)$

$\Rightarrow \exists\, t^* \in [0,1]$ such that $g'(t^*) = g(1) - g(0)$. This is

$f'(t^*x + (1-t^*)y) \cdot (x-y) = f(x) - f(y)$

Now set $h = x-y$. The condition the problem says $df * h = 0$

$\Rightarrow f(x) = f(y)$. Contradiction.

3. As hinted Construct $g = (f(x,y), y)$. $Dg = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 0 & 1 \end{pmatrix}$ suppose $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$ then there is a neighborhood U of (x_0, y_0), such that $g|_U$ is bijective $g: \mathbb{R}^2 \to \mathbb{R}^2$, The image of U under g is also an open set. Then there must be two points in the image with the same first coordinate. This means $f(x,y)$ is not injective.

4. $Df = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$ $\det(Df) = 4(x^2+y^2) \neq 0$ in U.

f is not diffeo. (x, y) and $(-x, -y)$ are mapped to the same image. Not inject

5. $f: \mathbb{R}^2 \to \mathbb{R}$ is bilinear

for fixed y, f is a linear function of x

Then $f(x,y) = g(y)\, x$.

For fixed x. f is a linear function of y

$\Rightarrow f(x,y) = cyx$, where $c = f(1,1)$

$df = cy\,dx + cx\,dy$

6. To show f is regular we need to show $f(x,y) = (x, y, x^2 - y^2)$ is differentiable homeo and $d_x f$ injective. f is written as graph over $\mathbb{R}^2$. So it is homeo.

$$d_x f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & -2y \end{pmatrix} \text{ injective} \Rightarrow f \text{ is regular}$$

We can check $(u+v)^2 - (u-v)^2 = 4uv$, $f(u,v) \to \begin{pmatrix} u+v \\ u-v \\ 4uv \end{pmatrix}$ is bijective on S

the map from $(u,v) \to (x,y)$ give by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ is an isometry on $\mathbb{R}^2$

This shows $f(u,v)$ is a parametrization

7. The ~~surface~~ Cylinder is give by $F(x,y,z) = x^2 + y^2 - 1 = 0$, 0 is a regular value of F. This shows the cylinder is regular.

A parametrization is given by

$$f(\theta, z) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \quad \begin{matrix} \theta \in [0, 2\pi) \\ z \in \mathbb{R} \end{matrix}$$

this parametrization is bijective and $df = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix}$ injective

8. $df = (0, 0, 2z)$ so $f(x,y,z) = 0 \Rightarrow z = 0$,

$\Rightarrow df = (0,0,0)$. 0 is not a regular value

$f^{-1}(0) = \{(x,y) \mid (x,y) \in \mathbb{R}^2\}$ $\mathbb{R}^2$ is a regular surface.