1. $F(x,y,z) = (x+y+z-1)^2$

$DF(x,y,z) = 2(x+y+z-1)\,(1,1,1)$

$DF = 0 \Leftrightarrow x+y+z=1$ This is the set of critical points

If c is a regular value the $F^{-1}(c)$ is regular surface.

For $c=0$, the critical value, $F^{-1}(0) = \{(x,y,z) \mid x+y+z=1\}$ is a hyperplane in $\mathbb{R}^3$, which is also a regular surface

2. (a) regular Define $F(x,y,z) = x^3+y^3+z^3-3xyz$

$DF = 3(x^2-yz,\ y^2-xz,\ z^2-xy)$

At a critical point (x,y,z), we should have $x^2=yz$, $y^2=xz$, $z^2=xy$

$\Rightarrow (x-y)^2+(y-z)^2+(x-z)^2=0 \Rightarrow x=y=z$, The corresponding critical value is

So 1 is a regular value $F^{-1}(1)$ is regular surface

(b) not regular

Consider the z-axis $\{x=0, y=0\}$. Restricted in a neighborhood of z-axis

S is not a manifold

3. The fact that a torus is a surface of revolution can be seen as follows: consider a circle of radius 1 centered at $(0, a, 0)$ in the y-z plane, $a>1$

Then rotate the circle around z-axis for 2π, we get a torus

We have parametrization ~~(a cos θ, a sin~~ $((a+\cos\theta)\cos\phi,\ (a+\cos\theta)\sin\phi,\ \sin$

if, $\theta\in[0,2\pi)$, $\phi\in[0,2\pi)$, then the curves $((a+1)\cos\phi,\ (a+1)\sin\phi,\ 0)$ and $((a+\cos\theta),\ 0,\ \sin\theta)$ need other parametrizations to cover.

so we use a second cover $\theta\in[-\pi,\pi)$ $\phi\in[-\pi,\pi)$

4. $Dg = 2(x,y,z)$ We form the function $F(x,y,z) = (g,h) : \mathbb{R}^3 \to \mathbb{R}^2$

$Dh = (1,-1,-1)$ Then $DF = \begin{pmatrix} 2x & 2y & 2z \\ 1 & -1 & -1 \end{pmatrix}$, this matrix always has rank 2

The reason is x,y,z cannot be zero simultaneously since $g=0$, and (x,y,z) cannot be parallel to $(1,-1,-1)$ since $h=0$. Therefore $(0,0)$ is a regular value of F.

$F^{-1}(0,0)$ is a manifold of dimension 1

5. The paraboloid is $(x, y, \cancel{2}x^2+y^2)$

The plane is $(x, y, 0)$.

Therefore the paraboloid parametrized by (x, y) is homeomorphic to the plane. Now we need to show the differentiability

~~d~~ $\cancel{DZ = (2x}$

Consider the ~~dust~~ map $P: (x, y, 0) \to (x, y, x^2+y^2)$

$$DF = \begin{pmatrix} 1 & 0 & 2x \\ 0 & 1 & 2y \end{pmatrix} \quad \text{differentiable}$$

On the other hand $\pi: (x, y, x^2+y^2) \to (x, y, 0)$ is a projection to the first two components. which is also differentiable

These show the paraboloid is diffeo to a plane

6. Any curve $\gamma \subset \{x^2+y^2-z^2=1\}$, $\gamma = (x(t), y(t), z(t))$ gives us

$$2x\,x'(t) + 2y\,y'(t) - 2z\,z'(t) = 0$$

We know $z=0$. This shows $\cancel{(x', y')}$ $xx'+yy'=0$. This means (x', y') is tangent to the circle $\{x^2+y^2=1, z=0\}$. We also have z' can be arbitrary.

This implies the tangent plane at point $(x, y, 0)$ is spanned by the vector tangent to $\{x^2+y^2=1, z=0\}$ and the vector $(0, 0, 1)$. Such tangent plane is always parallel to z-axis

7. It's easy to see A is homeomorphism, A^{-1} has the same form. $DA = Id$.

So A is a diffeomorphism.

8. Consider a chart $\phi: \mathbb{R}^2 \to S$. Then we have $f \circ \phi: \mathbb{R}^2 \to \mathbb{R}$

$f(p) = |p - p_0|^2$ as a ~~function~~ quadratic function of p is differentiable.

$f \circ \phi$ is also differentiable since ϕ is a

Consider a curve γ lying in the image of ϕ. $\gamma(0) = p$, $\gamma'(0) = w \in T_pS$, $\gamma: (-\delta, \delta)$

Then $f(\gamma(t)) = |\gamma(t) - p_0|^2$ as a function of t is defined on an open interval $(-\delta, \delta)$. then

$$\left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = df_p\,\gamma'(0) = df_p(w) = 2(p-p_0)\cdot\gamma'(0)$$

$$= 2w\cdot(p-p_0)$$