Spring 2012 - Math 437 Section 0101

Homework #4 - **Due February 28th**

1. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and define the (Hamiltonian) function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by
$$H(x,y) = \frac{1}{2}|y|^2 + V(x)$$
   (a) Prove that c is a regular value of H if and only if it is a regular value of V.
   (b) Let c be a regular value of H and $M = H^{-1}(c)$. Show that
$$T_{(x,y)}M = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \,;\, \langle y, \eta\rangle + \langle \nabla V(x), \xi\rangle = 0\}$$
   (where $\langle x, y\rangle$ denotes the inner product (or dot product) of two vectors in $\mathbb{R}^n$)

2. Let S be the graph of a smooth function g, $S = \{(x,y,z)\,;\, z = g(x,y)\}$ and let $p_0 = (x_0, y_0, z_0) \in S$. Show that $T_{p_0}S$ is equal to the graph of $dg_{(x_0,y_0)}$.

3. Let $\mathbb{S}^1$ denotes the unite circle $\{(x,y) \in \mathbb{R}^2\,;\, x^2 + y^2 = 1\}$. Let $N = (0,1)$ and $S = (0,-1)$ denote the north and south pole on $\mathbb{S}^1$. We define the stereographic projection
$$\varphi_1 : \mathbb{S}^1 \setminus \{N\} \to \mathbb{R}$$
   where $\varphi_1(p)$ is the x-coordinate of the intersection of the line passing through p and N with the x-axis. Similarly, we dedine
$$\varphi_2 : \mathbb{S}^1 \setminus \{S\} \to \mathbb{R}$$
   where $\varphi_2(p)$ is the x-coordinate of the intersection of the line passing through p and S with the x-axis.

   Show that $\{(\mathbb{R}, \varphi_1^{-1}), (\mathbb{R}, \varphi_2^{-1})\}$ defines a differential structure on $\mathbb{S}^1$.

4. We say that two surfaces S_1 and S_2 in $\mathbb{R}^3$ intersect *transversally* if whenever $p \in S_1 \cap S_2$ then $T_pS_1 \neq T_p(S_2)$.

   Assume that S_1 and S_2 are defined as level sets of smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ and $G : \mathbb{R}^3 \to \mathbb{R}$ (for instance, $S_1 = F^{-1}(0)$ and $S_2 = G^{-1}(0)$ where 0 is a regular value of F and G). Prove that if S_1 and S_2 intersect *transversally*, then $S_1 \cap S_2$ is a regular curve (or a 1-dimensional manifold).

5. Let M be a n-dimensional manifold with a given differential structure $\{(U_\alpha, f_\alpha)\}$. Let V be an open subset of M. Show that V is a n-dimension manifold.

6. Let M be a connected n-manifold (here connected means that for any two points p_1 and p_2 on M, we can find a smooth curve $\alpha : [0,1] \to M$ such that $\alpha(0) = p_1$, $\alpha(1) = p_2$). Let $f : M \to \mathbb{R}$ be a differentiable function on M such that $df_p(v) = 0$ for all $p \in M$ and $v \in T_pM$. Show that f is constant on M.