Hw 4

1. (a) $\nabla H = (\nabla V(x), y)$

c is a regular value of H. iff for $\forall (x,y) \in H^{-1}(c)$, (x,y) is not a critical

consider the point $(x,0) \in H^{-1}(c)$ i.e. $(x,0) \in V^{-1}(c)$

∇H is nondegenerate iff $\nabla V(x)$ is nondegenerate.

This implies. c is a regular value of H iff it is a regular value of $V(x)$

(b). consider a curve $\gamma: (-\varepsilon, \varepsilon) \to M = H^{-1}(c)$ $\gamma(0) = (x,y)$, $\gamma(t) = (x(t), y(t))$

$\gamma'(0) = (\xi, \eta) \in T_{(x,y)}M$ then $\frac{1}{2}(y(t))^2 + V(x(t)) = c$

$\frac{1}{2} y(t) \cdot y(t) + V'(x) \cdot x'(t) = 0$, evaluated at $t=0$, gives us

$\langle y, \eta \rangle + \langle \nabla V(x), \xi \rangle = 0$

2. $S = \{(x,y,z): z = g(x,y)\}$

$dz = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy$

$T_{p_0}S$ is a plane tangent to S at the point p_0.

Therefore $T_{p_0}S$ has the express-ion $z_0 + \frac{\partial g}{\partial x}\Big|_{p_0} (x - x_0) + \frac{\partial g}{\partial y}\Big|_{p_0} (y - y_0)$

this is also the expression of the graph of $dg_{(x_0, y_0)}$

3. The expressions φ_1, φ_2 can be written down explicitly

$\varphi_1(x,y) = \frac{x}{1-y}$, $\varphi_2(x,y) = \frac{x}{1+y}$

It's a direct computation that both $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are differentiable from $\mathbb{R}$ to $\mathbb{R}$, $\varphi_1^{-1}(\mathbb{R}) \cup \varphi_2^{-1}(\mathbb{R})$ covers S^1.

This shows $(\mathbb{R}, \varphi_1^{-1})$, and $(\mathbb{R}, \varphi_2^{-1})$ define a differential structure on S^1

4. consider ~~the~~ S defined as $S = S_1 \cap S_2$.
Then S has the expression

$$\{ F=0, G=0 \}$$

The derivative is $\begin{pmatrix} dF \\ dG \end{pmatrix}$. We know from problem 1(b)

$d_pF \perp T_pS_1$, $d_pG \perp T_pS_2$. Then $T_pS_1 \neq T_p(S_2)$ implies d_pF, d_pG are not colinear for any $p \in S_1 \cap S_2$. This implies $\begin{pmatrix} dF \\ dG \end{pmatrix}$ nondegenerate
Implicit function theorem shows $S_1 \cap S_2$ is a regular curve.

5. Consider $V_\alpha = U_\alpha \cap f_\alpha^{-1}(U)$, Then $\bigcup_\alpha V_\alpha$ covers V. V is open in M. it must have $\dim V = n$
Moreover, $\{(V_\alpha, f_\alpha|_{V_\alpha})\}$ defines a differential structure on V.
This implies V is a n-dim manifold

6 Suppose f is not a constant on M, then there must be two points
$p_1, p_2 \in M$ s.t. $f(p_1) \neq f(p_2)$
Find a smooth curve $\alpha : [0,1] \to M$ s.t. $\alpha(0) = p_1$, $\alpha(1) = p_2$
Then $f(\alpha(t))$ is not a constant function on $[0,1]$
~~So then~~ But $\frac{d}{dt}\left(f(\alpha(t))\right) = df \cdot \alpha'(t) = 0$ since $df_p(v) = 0$ $\forall p \in M, v \in T_p$
This is a contradiction.
So f is constant on M