Spring 2012 - Math 437 Section 0101

Homework #5 - **Due March 13**

1. Let $M = \{(x, y, z)\,;\, xy^2 + yz^2 + zx^2 = 1\}$
   (a) Show that M is a regular surface in $\mathbb{R}^3$.
   (b) Define $\pi : M \to \mathbb{R}$ by $\pi(x, y, z) = x$. Find the critical points and critical values of π.

2. Consider the subset $M = \{(x, y) \in \mathbb{R}^2\,;\, x^2 + y^2 = 1\} \cup \{(x, 0) \,|\, 1 < x < 2\}$.
   (a) Sketch M and explain why M is not a regular curve (or submanifold) in $\mathbb{R}^2$.
   (b) Prove that the function $f : (0, 2) \to M$ given by $f(s) = (\cos(2\pi s), \sin(2\pi s))$ if $0 < s < 1$ and $f(s) = (s, 0)$, if $1 \leq s < 2$ defines a differential structure on M.

3. Let $\alpha \in \Lambda^p(\mathbb{R}^{n*})$.
   Prove that if $\{v_1, \ldots, v_p\}$ is linearly dependent, then $\alpha(v_1, \ldots, v_p) = 0$.

4. Consider the forms $\alpha = x\,dx - y\,dy$, $\beta = z\,dx \wedge dy + x\,dy \wedge dz$ and $\gamma = z\,dy$. Compute $\alpha \wedge \beta$, $\alpha \wedge \gamma$ and $\beta \wedge \gamma$.

5. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\phi(x_1, x_2, x_3) = (x_1x_2, x_1x_3, x_2x_3)$. Find
   (a) $\phi^* dy_1$, $\phi^* dy_2$, $\phi^* dy_3$
   (b) $\phi^*(dy_1 \wedge dy_2)$
   (c) $\phi^*(dy_1 \wedge dy_2 \wedge dy_3)$

   (we denote by (y_1, y_2, y_3) the coordinates in $\phi(\mathbb{R}^3) \subset \mathbb{R}^3$).

6. Let $f(r, \theta, \varphi) = (r \cos\theta \sin\varphi, r \sin\theta \sin\varphi, r \cos\varphi)$ be the spherical coordinates in $\mathbb{R}^3$. Calculate $f^*\alpha$ for the following differential forms α:

$$dx,\ dy,\ dz,\ dx \wedge dy,\ dx \wedge dz,\ dy \wedge dz,\ dx \wedge dy \wedge dz.$$