HW 5

1. (a) define $F = xy^2 + yz^2 + zx^2$

Then $M = F^{-1}(1)$

$DF = (y^2 + 2zx, 2xy + z^2, 2yz + x^2)$

We know $\begin{cases} y^2 + 2zx = 0 \\ 2xy + z^2 = 0 \\ 2yz + x^2 = 0 \end{cases}$

has only solutions $x = y = z = 0$ But $F(0,0,0) = 0$

So the point $(x,y,z) = (0,0,0)$ does not lie on the manifold M,

Using Implicit function theorem, we see M is regular

(b) We need to solve x as a function of (y, z)

The critical point is the point (y, z) such that

$$\begin{cases} \frac{\partial x}{\partial y}(y,z) = 0 \\ \frac{\partial x}{\partial z}(y,z) = 0 \end{cases}$$

Using Implicit function theorem, we need to ensure

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

This gives $\begin{cases} 2xy + z^2 = 0 \\ 2yz + x^2 = 0 \end{cases}$

From the second equation $z = -\frac{x^2}{2y}$

$\Rightarrow 2xy + \left(\frac{x^2}{2y}\right)^2 = 0 \Rightarrow 8y^3 = -x^3 \Rightarrow x = -2y$

$\Rightarrow z = -2y$

We plug this back to $xy^2 + yz^2 + zx^2 = 1 \Rightarrow$

$-2y^3 + 4y^3 - 8y^3 = 1 \Rightarrow -6y^3 = 1 \Rightarrow y = -\frac{1}{6^{\frac{1}{3}}}$

$\Rightarrow (x,y,z) = \left(\frac{2}{6^{\frac{1}{3}}}, -\frac{1}{6^{\frac{1}{3}}}, \frac{2}{6^{\frac{1}{3}}}\right)$ critical point

critical value is $\frac{2}{6^{\frac{1}{3}}}$

2 (a) The subset M contains the point $(1,0)$. There's no manifold structure at that point. So M is not a regular curve

(b) I do not find definition of differential structure in the textbook. But using wiki I find the charts should be defined on open sets. Then $f: (0,2) \to M$ can be considered as only one chart. At the point $f(1) = (1,0)$, f is not differentiable. So it can only be considered as a C^0 differentiable structure.

I give full marks if one shows injecti

3. If $\{v_1, \ldots, v_p\}$ is linearly dependent, we write $v_p = \sum_{i=1}^{p-1} a_i v_i$

$$\Rightarrow \alpha(v_1, \ldots, v_p) = \sum_{i=1}^{p-1} a_i \alpha(v_1, \ldots, v_i)$$

using the alternating property of α each of the summands is 0

$\Rightarrow \alpha(v_1, \ldots, v_p) = 0$

4. $\alpha\wedge\beta = x^2\,dx\wedge dy\wedge dz$

$\alpha\wedge\gamma = xz\,dx\wedge dy$

$\beta\wedge\gamma = 0$

5. $\phi^* dy_1 = x_2\,dx_1 + x_1\,dx_2$

$\phi^* dy_2 = x_3\,dx_1 + x_1\,dx_3$

$\phi^* dy_3 = x_2\,dx_3 + x_3\,dx_2$

$\phi^*(dy_1\wedge dy_2) = \phi^* dy_1 \wedge \phi^* dy_2$

$= x_1x_2\,dx_1\wedge dx_3 - x_1x_3\,dx_1\wedge dx_2 + x_1^2\,dx_2\wedge dx_3$

$\phi^*(dy_1\wedge dy_2\wedge dy_3) = -2x_1x_2x_3\,dx_1\wedge dx_2\wedge dx_3$

6. $f^* dx = \cos\theta\sin\phi\,dr - r\sin\theta\sin\phi\,d\theta + r\cos\theta\cos\phi\,d\phi$

$f^* dy = \sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi$

$f^* dz = \cos\phi\,dr - r\sin\phi\,d\phi$

$f^*(dx\wedge dy) = r\sin^2\phi\,dr\wedge d\theta - r^2\sin\phi\cos\phi\,d\theta\wedge d\phi$

$f^*(dx\wedge dz) = -r\cos\theta\,dr\wedge d\phi + r\sin\theta\sin\phi\cos\phi\,dr\wedge d\theta$

$+ r^2\sin\theta\sin^2\phi\,d\theta\wedge d\phi$

$f^*(dy\wedge dz) = -r\sin\theta\,dr\wedge d\phi - r\cos\theta\sin\phi\cos\phi\,dr\wedge d\theta$

$- r^2\cos\theta\sin^2\phi\,d\theta\wedge d\phi$

$f^*(dx\wedge dy\wedge dz) = r^2\sin\phi\,dr\wedge d\phi\wedge d\theta.$