

Spring 2012 - Math 437 Section 0101

Homework #6 - Due March 27

1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$f(s, t) = (\cos(s) \sin(t), \sqrt{s^2 + t^2}, st)$$

and $\omega = dx \wedge dz$. Compute $f^*\omega$.

2. Let $\omega = xydx \wedge dz + zdx \wedge dy$ be a differential 2-form on \mathbb{R}^3 . Compute the exterior derivative $d\omega$.

3. Which of the following forms on \mathbb{R}^3 are closed?

- (a) $\omega = xdx \wedge dy \wedge dz$
- (b) $\omega = zdy \wedge dx + xdy \wedge dz$
- (c) $\omega = xdx + ydy$

4. Which of the following forms ω on \mathbb{R}^2 are exact? For each exact form, find σ such that $\omega = d\sigma$.

- (a) $\omega = xdy - ydx$
- (b) $\omega = xdy + ydx$
- (c) $\omega = dx \wedge dy$

5. Let f and g be real-valued functions on \mathbb{R}^2 . Prove that

$$df \wedge dg = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dx \wedge dy.$$

6. (a) Let α and β be two closed forms. Prove that $\alpha \wedge \beta$ is also closed.

(b) Let α be closed and β exact. Prove that $\alpha \wedge \beta$ is exact.

7. Let $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ and $Y = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ be two vector fields in \mathbb{R}^2 (where $a, b \in \mathbb{R}$ are constant). Compute $[X, Y]$.

8. Let $X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ be a vector field in \mathbb{R}^2 (where $a, b \in \mathbb{R}$ are constant). Find the flow ϕ_t for X .

9. Let $[a, b] \in \mathbb{R}$ be an interval. Let $\omega \in \Omega^1(\mathbb{R})$ be a smooth 1-form. ω can be written as

$$\omega(t) = g(t)dt$$

with $g : \mathbb{R} \rightarrow \mathbb{R}$ smooth function. We define

$$\int_{[a,b]} \omega = \int_a^b g(t) dt.$$

- (a) Let $r : [c, d] \rightarrow [a, b]$ be a smooth map such that $r(c) = a$ and $r(d) = b$. Calculate $r^*\omega$ and show that

$$\int_{[c,d]} r^*\omega = \int_{[a,b]} \omega.$$

Let now M be a smooth manifold and let $\omega \in \Omega^1(M)$ be a smooth 1-form on M . For any smooth curve $\gamma : [a, b] \rightarrow M$, we define

$$\int_{\gamma} \omega = \int_{[a,b]} \gamma^*\omega.$$

- (b) Let $\bar{\gamma}(s) = \gamma(r(s))$ be a reparametrization of γ (with r as in (a)). Show that

$$\int_{\gamma} \omega = \int_{\bar{\gamma}} \omega.$$

- (c) Assume that ω is exact, that is there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that $\omega = df$. Show that

$$\int_{\gamma} \omega = f(\gamma(b)) - f(\gamma(a)).$$