1. $f^* dx = d(\cos s \sin t) = -\sin s \sin t \, ds + \cos s \cos t \, dt$

$f^* dz = d st = t\,ds + s\,dt$

$f^*(dx \wedge dz) = (-s \sin s \sin t - t \cos s \cos t) \, ds \wedge dt$

2. $d\omega = d(xy\, dx\wedge dz + z\, dx \wedge dy)$

$= x\, dy \wedge dx \wedge dz + dz \wedge dx \wedge dy$

$= (-x+1)\, dx \wedge dy \wedge dz$

3. (a) $d\omega = dx \wedge (dx \wedge dy \wedge dz) = 0$ closed

(b) $d\omega = dz \wedge dy \wedge dx + dx \wedge dy \wedge dz = 0$ closed

(c) $d\omega = dx \wedge dx + dy \wedge dy = 0$ closed

4. (a) $d\omega = dx \wedge dy - dy \wedge dx = 2\, dx \wedge dy \neq 0$ ω is not closed, so not exact

(b) $\omega = d(xy)$ so $\sigma = xy$, $\omega = d\sigma$. exact

(c) $\omega = dx \wedge dy$ take $\sigma = x\,dy \Rightarrow \omega = d\sigma$ exact

5. $df \wedge dg = \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \wedge \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy\right) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} dx \wedge dy + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} dy \wedge dx$

$= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dx \wedge dy$

6. (a) $d(\alpha\beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$. suppose α is a k form We know $d\alpha = d\beta = 0$. This implies $d(\alpha\beta) = 0$.

(b) suppose $\beta = d\gamma$ Then $\alpha\beta = \alpha\, d\gamma = (-1)^k d(\alpha\gamma)$ since $d\alpha = 0$, (suppose α is k-form)

$= d((-1)^k \alpha\gamma)$

7. $[X, Y] = XY - YX = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$= \left(ax\frac{\partial^2}{\partial x\partial y} + bx\frac{\partial^2}{\partial y^2} - ay\frac{\partial^2}{\partial x^2} - by\frac{\partial^2}{\partial x\partial y}\right) - \left(ax\frac{\partial^2}{\partial x\partial y} + a\frac{\partial}{\partial y} - ay\frac{\partial^2}{\partial x^2} + bx\frac{\partial^2}{\partial y^2} - b\frac{\partial}{\partial x} - by\frac{\partial^2}{\partial x\partial y}\right)$

$= -a\frac{\partial}{\partial y} + b\frac{\partial}{\partial x}$

8. Set $\phi_t = (x(t), y(t))$

$\Rightarrow \dot{\phi}_t = X \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} at \\ bt \end{pmatrix}$

9. $\omega = g(t)dt$

(a) $r^*\omega = g(r(s))\, d(r(s)) = g(r(s))\, r'(s)\, ds$

$\int_{[c,d]} r^*\omega = \int_{[a,b]} \omega$ is the change of variable formula

(b). $\int_{\tilde{\gamma}} \omega = \int_{[c,d]} \tilde{\gamma}^* \omega = \int_{[c,d]} (\gamma \circ r)^* \omega = \int_{[c,d]} r^*(\gamma^*\omega) = \int_{[a,b]} \gamma^*\omega = \int_{\gamma} \omega$

(c) $\int_{\gamma} \omega = \int_{[a,b]} \gamma^* \omega = \int_{[a,b]} \gamma^* df = \int_{[a,b]} d f(\gamma) = f(\gamma(b)) - f(\gamma(a))$