Spring 2012 - Math 437 Section 0101

Homework #7 - **Due April 3rd**

1. (a) Let $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$ be defined by
$$\omega(x,y) = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
Prove that ω is a closed form.

   (b) Let $\gamma : [0,2\pi] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the smooth curve $\gamma(t) = (\cos t, \sin t)$. Compute
$$\int_\gamma \omega.$$
(we recall (see homework 6) that for any smooth curve $\gamma : [a,b] \to M$, we have $\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega$.)

   (c) Explain why this implies that ω cannot be exact (use Homework 6, Problem 9 (c)).

2. Let $\omega = x_3\,dx_1 \wedge dx_2 + x_1\,dx_2 \wedge dx_3 \in \Omega^2(\mathbb{R}^3)$.

   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
$$f(u,v) = (u^2, uv, v^2).$$
Compute $f^*\omega$ (write your result in the form $f^*\omega = g(u,v)du \wedge dv$).

   (b) Same question with $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(u,v) = (1, v\sin u, u\cos v)$.

3. Let $M = \{(x,y,z) \in \mathbb{R}^3\,;\, z = x^2+y^2,\ z < 1\}$ be a smooth 2-manifold in $\mathbb{R}^3$ (M is an open subset of the paraboloid).

   (a) Show that M is orientable.

   (b) Let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3)$. Compute
$$\int_M \omega.$$

4. Let S be a regular surface in $\mathbb{R}^3$. We want to show that S is orientable **if and only if** there exists a smooth map $n : S \to \mathbb{R}^3$ such that for all $p \in S$, we have $||n(p)|| = 1$ and $n(p)$ orthogonal to T_pS (we say that $n(p)$ is a unit normal vector to S at p).

(a) First, assume that such a smooth map n exists. Let $\omega_0 \in \Lambda^3(\mathbb{R}^{3*})$ be defined by $\omega_0 = dx_1 \wedge dx_2 \wedge dx_3$. Show that the interior product

$$\omega(p) = i_{n(p)}\omega_0$$

defines a differential 2-form on S such that $\omega(p) \neq 0$ for all $p \in S$ and that this implies that S is orientable.

(b) Assume now that S is orientable. Let p be a point on S, and assume that $p = f_\alpha(u_0, v_0)$ for some parametrization f_α. We define the cross-product

$$N_\alpha(p) = \frac{\partial f_\alpha}{\partial u}(u_0, v_0) \times \frac{\partial f_\alpha}{\partial v}(u_0, v_0) \in \mathbb{R}^3.$$

(i) Explain why $N_\alpha(p)$ is orthogonal to T_pS, and why $N_\alpha(p) \neq 0$.

(ii) We now set

$$n(p) = \frac{N_\alpha(p)}{||N_\alpha(p)||}.$$

Show that this defines a smooth map $n : f_\alpha(U_\alpha) \to \mathbb{R}^3$ with $n(p)$ normal unit vector (first show that $p \mapsto N_\alpha(p)$ is a smooth map).

In order to conclude, it remains to show that this construction does not depend on the parametrization. Assume that $p \in f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$ for two parametrizations f_α and f_β defining the same orientation (that is $\det(J(f_\beta^{-1} \circ f_\alpha)) > 0$). The construction above yields two vectors $N_\alpha(p)$ and $N_\beta(p)$ (depending on whether we use f_α or f_β in the cross product).

(iii) Show that $N_\beta(p) = \lambda N_\alpha(p)$ for some $\lambda > 0$ (the positivity of λ is very important here) and that

$$n(p) = \frac{N_\alpha(p)}{||N_\alpha(p)||} = \frac{N_\beta(p)}{||N_\beta(p)||}.$$

(Hint: Use the fact, proved in class, that the matrix $P = J(f_\beta^{-1} \circ f_\alpha)$ is the change of coordinate matrix from the basis $\{\frac{\partial f_\alpha}{\partial u}, \frac{\partial f_\alpha}{\partial v}\}$ to the basis $\{\frac{\partial f_\beta}{\partial u}, \frac{\partial f_\beta}{\partial v}\}$)

5. Using the result of Problem 4, show that if S is a regular surface of $\mathbb{R}^3$ defined by $S = F^{-1}(c)$ where c is a regular value of a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, then S is orientable.