Spring 2012 - Math 437 Section 0101

Homework #8 - **Due April 10th**

1. Let $M = \{(x, y, z) \in \mathbb{R}^3 ;\, z = x^2 + y^2,\ z \leq 1\}$ (M is a differential 2-manifold with boundary). Let $\omega = y^2 dx + (x + y)dy + dz$. Compute

$$\int_M d\omega$$

in two different ways: First by a direct computation, computing $d\omega$ and using a parametrization of M to pull-back $d\omega$ to a subset of $\mathbb{R}^2$. Then by using Stokes theorem.

2. Let

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

Compute $\int_{\mathbb{S}^2} \omega$ in two different ways:

(a) By using Stokes theorem and the fact that $\mathbb{S}^2 = \partial B_1(0)$.

(b) By a direct computation using the parametrization

$$f(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi),$$

with $(\theta, \varphi) \in U = (0, 2\pi) \times (0, \pi)$. Note that $f(U) \neq \mathbb{S}^2$, since half a circle is missing, but you can use the fact that

$$\int_{f(U)} \omega = \pm \int_{\mathbb{S}^2} \omega$$

without justification.

Remark: ω is actually the area form on $\mathbb{S}^2$, so you should find $\int_{\mathbb{S}^2} \omega$ =area of $\mathbb{S}^2$.

3. Let

$$\omega = f_1(x)dx_2 \wedge dx_3 + f_2(x)dx_3 \wedge dx_1 + f_3(x)dx_1 \wedge dx_2$$

be a differential 2-form in $\mathbb{R}^3$.

(a) Let $F = (f_1, f_2, f_3)$. Show that

$$d\omega = \operatorname{div} F dx_1 \wedge dx_2 \wedge dx_3$$

(where we recall that $\operatorname{div} F = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i}$).

(b) Deduce the **divergence theorem**: If U is an open subset of $\mathbb{R}^3$ with smooth boundary and F is a smooth vector field in $\mathbb{R}^3$, then

$$\int_U \operatorname{div} F = \int_{\partial U} F \cdot n \, dA$$

where n is the outward unit normal vector on the regular surface ∂U, and dA is the area form on dA (as defined in class).

4. Prove the classical Stokes theorem: Let S be a compact oriented 2-manifold in $\mathbb{R}^3$ with boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field of $\mathbb{R}^3$ defined in a neighborhood of S. Show that

$$\int_S (\text{curl } F) \cdot n \, dA = \int_{\partial S} f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

where n is a unit normal vector to S, and curl F is defined by

$$\text{curl } F = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right)$$

5. (a) Let $\omega = xdy - ydx$, U a open subset of $\mathbb{R}^2$ with smooth boundary ∂U. Show that the area of U is equal to

$$\frac{1}{2} \int_{\partial U} \omega.$$

(b) Let $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$, U a open subset of $\mathbb{R}^3$ with smooth boundary ∂U. Show that the area of U is equal to

$$\frac{1}{3} \int_{\partial U} \omega.$$

6. Suppose that $N = \partial M$, where M is a compact n-manifold with boundary. Let $f : N \to M$ be a smooth map and ω be a **closed** $(n-1)$-form on M. Prove that if f can be extended into a smooth map $\bar{f} : M \to M$, then $\int_N f^*\omega = 0$.