HW 8

1. Method 1. $f^* dw = 2y\, dy \wedge dx + dx \wedge dy + d(x^2+y^2) = (1-2y) dx \wedge dy$

where $f: \mathbb{R}^2 \to M$, $(x,y) \mapsto (x, y, z = x^2+y^2)$

Then $\int_M dw = \int_{x^2+y^2 \le 1} f^* w = \int_{x^2+y^2\le 1} (1-2y) dx \wedge dy \underset{\uparrow}{=} \int_{x^2+y^2 \le 1} dx \wedge dy \overset{\text{Area of disk}}{=} \pi$

y is odd function defined on disk.

Method 2, use Stokes

$\int_M dw = \int_{\partial M} w = \int_{x^2+y^2=1} f^* w = \int_{x^2+y^2=1} y^2 dx + (x+y) dy + d(x^2+y^2) = \int_{x^2+y^2=1} (2x+y^2) dx + (x+3y) dy$

We use parametrization $\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow \int_0^{2\pi} -(2\cos t + \sin^2 t) \sin t\, dt + (\cos t + 3 \sin t) \cos t\, dt$

$= \int_0^{2\pi} \sin t \cos t - \sin^3 t + \cos^2 t\, dt = \pi$

2. (a) $\int_{S^2} w = \int_{\partial B_1(0)} w = \int_{B_1(0)} dw = \int_{B_1(0)} 3\, dx \wedge dy \wedge dz = 3 \operatorname{vol}(B_1(0)) = 4\pi$

(b) $f^* w = \cos\theta \sin\varphi\, d(\sin\theta \sin\varphi) \wedge d\cos\varphi$
$+ \sin\theta \sin\varphi\, d\cos\varphi \wedge d(\cos\theta \cos\varphi)$
$+ \cos\varphi\, d(\cos\theta \sin\varphi) \wedge d(\sin\theta \sin\varphi)$

After computation this gives $f^* w = \sin\varphi\, d\varphi \wedge d\theta$

$\int_{S^2} w = \int_U f^* w = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \sin\varphi = 4\pi$

3. (a) $dw = df_1 \wedge dx_2 \wedge dx_3 + df_2 \wedge dx_3 \wedge dx_1 + df_3\, dx_1 \wedge dx_2$

use the expansion $df_i = \frac{\partial f_i}{\partial x_1} dx_1 + \frac{\partial f_i}{\partial x_2} dx_2 + \frac{\partial f_i}{\partial x_3} dx_3$ to obtain

$dw = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 = \operatorname{div} F\, dx_1 \wedge dx_2 \wedge dx_3$

(b) According to the Stokes theorem

$\int_U \operatorname{div} F = \int_{\partial U} f_3\, dx_1 \wedge dx_2 + f_2\, dx_3 \wedge dx_1 + f_1\, dx_2 \wedge dx_3$

~~The area 2-form dA is a 2-form~~

The area 2-form dA is a 2 form defined as $dA(v_1,v_2)=\vec{n}\cdot(v_1\times v_2)$, $v_1,v_2\in T(\partial U)$

Suppose $\vec{n}=(n_1,n_2,n_3)$, using cosine theorem, to project $T(\partial U)$ to 3 coordinate planes to obtain $n_1\,dA=dx_2\wedge dx_3$, $n_2\,dA=dx_3\wedge dx_1$, $n_3\,dA=dx_1\wedge dx_2$.

Therefore $f_3\,dx_1\wedge dx_2+f_2\,dx_3\wedge dx_1+f_1\,dx_2\wedge dx_3=F\cdot n\,dA$.

4. Using stokes, we get $\int_{\partial S} f_1dx_1+f_2dx_2+f_3dx_3=\int_S df_1\wedge dx_1+df_2\wedge dx_2+df_3\wedge dx_3$

$$=\int_S\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)dx_2\wedge dx_3+\left(\frac{\partial f_3}{\partial x_1}-\frac{\partial f_1}{\partial x_3}\right)dx_3\wedge dx_1+\left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)dx_1\wedge dx_2$$

This can be written as $\int_S(\mathrm{curl}F\cdot n)\,dA$ using the same argument as problem 3.

5. $\mathrm{Area}(U)=\int_U dx\wedge dy=\frac{1}{2}\int_U d(xdy-ydx)=\frac{1}{2}\int_{\partial U}xdy-ydx=\frac{1}{2}\int_{\partial U}\omega$

$\mathrm{Volume}(U)=\int_U dx\wedge dy\wedge dz=\frac{1}{3}\int_U d(xdy\wedge dz-ydx\wedge dz+zdx\wedge dy)=\frac{1}{3}\int_U d\omega=\frac{1}{3}\int_{\partial U}\omega$

6. $\int_V f^*\omega=\int_{\partial M}f^*\omega=\int_{\partial M}\overline{f}^*\omega=\int_M d\overline{f}^*\omega=\int_M \overline{f}^*d\omega=0$