Spring 2012 - Math 437 Section 0101

Homework #9 - **Due April 24th**

1. Let M be a compact 3-manifold in $\mathbb{R}^3$ with boundary ∂M. We recall that ∂M is an orientable 2-manifold in $\mathbb{R}^3$, and so there exists a outward unit normal vector $n(p)$ defined on ∂M. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be two differentiable functions. Prove that (first Green's identity)

$$\int_M \text{grad } f \cdot \text{grad } g\, \omega + \int_M \Delta g\, \omega = \int_{\partial M} f(\text{grad } g \cdot n) dA$$

   where $\omega = dx_1 \wedge dx_2 \wedge dx_3$ is the usual volume element of $\mathbb{R}^3$ and dA is the area form on ∂M. Here, Δg denotes the laplacian of g and is defined by $\Delta g = \text{div grad } g = \sum_{i=1}^3 \frac{\partial^2 g}{\partial x_i^2}$.

   Hint: Define $H = f\text{grad } g : \mathbb{R}^3 \to \mathbb{R}^3$ and use the divergence theorem.

2. Let U be a star-shaped open subset of $\mathbb{R}^3$ (this means that there exists $x^0 \in U$ such that for all $x \in U$, the segment $[x^0, x] = \{tx^0 + (1-t)p\,;\, t \in [0,1]\}$ is contained in U).

   (a) Show that U is contractible.

   (b) Given $\omega = f(x)dx_1 \wedge dx_2 \wedge dx_3$ 3-form, define

$$\eta(x) = \left(\int_0^1 t^2 f\left(tx + (1-t)x^0\right) dt\right) \left[(x_1 - x_1^0)dx_2 \wedge dx_3 - (x_2 - x_2^0)dx_1 \wedge dx_3 + (x_3 - x_3^0)dx_1 \wedge dx_2\right].$$

   Show that

$$d\eta = \omega.$$

3. Let U be an open subset of $\mathbb{R}^n$ such that $M = \overline{U}$ is a differentiable manifold with boundary $\partial M \neq \emptyset$.

   (a) Let $\omega = x_1 dx_2 \wedge \cdots \wedge dx_n \in \Omega^{n-1}(M)$ and $\alpha = j^*\omega \in \Omega^{n-1}(\partial M)$ (where $j : \partial M \to M$ is the usual inclusion map $j(x) = x$). Show that

$$\int_{\partial M} \alpha > 0.$$

   (note that $d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$)).

   (b) Let $f : M \to \partial M$ be a smooth function such that $f(x) = x$ for all $x \in \partial M$. Show that $d(f^*\alpha) = 0$ and $j^*(f^*\alpha) = \alpha$.

   (c) Using (a) and (b), together with Stokes theorem, show that such a function f cannot exists.

(d) Deduce the following result (Brouwer fixed point theorem):

Theorem *Let B be the ball $\{x \in \mathbb{R}^n\,;\, ||x|| \leq 1\}$. If g is a smooth function $g : B \to B$, then there exists $x_0 \in B$ such that $g(x_0) = x_0$ (such a point is called a fixed point of g).*

Hint: Show that if g does not have a fixed point in B, then there exists a function $f : B \to \partial B$ such that $f(x) = x$ for all $x \in \partial D$. Such a function can be constructed as follows: Let $f(x)$ be the intersection of the half line starting in $g(x)$ and passing through x with ∂B. Find a formula for f to check that it is smooth and use (c) to conclude.

4. (a) Let M be a compact orientable n-manifold without boundary (i.e. $\partial M = \emptyset$) and let ω be a differential $(n-1)$-form on M. Show that there exists a point $p \in M$ such that $d\omega = 0$.

   (b) Using (a), show that there exists no immersion $f : \mathbb{S}^1 \to \mathbb{R}$ of the unit circle into the real line $\mathbb{R}$.

5. Let $F = (f_1, f_2, f_3)$ be a smooth vector field defined in an open subset U of $\mathbb{R}^3$. Assume that U is smoothly contractible to a point. Show that if $\operatorname{div} F = 0$ in U, then there exists a vector field $G = (g_1, g_2, g_3)$ such that $F = \operatorname{curl} G$.