1. Let $A = \begin{bmatrix} -1 & 2 & 0 \\ -4 & 5 & 0 \\ 2 & -2 & 1 \end{bmatrix}$.

   (a) Compute the characteristic polynomial of A and show that $\lambda = 1$ and $\lambda = 3$ are eigenvalues of A.

   (b) Diagonalize the matrix A.

2. (a) Show that the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonalizable.

   (b) Show that if $n \times n$ matrices A and B are similar, then A^2 and B^2 are similar too.

3. (a) Find the quadratic form corresponding to the symmetric matrix $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 7 \end{bmatrix}$.

   (b) Classify the quadratic form $Q(\vec{x}) = 2x_1^2 + 10x_1x_2 + 2x_2^2$ as positive definite, negative definite or indefinite.

   (c) Give P and D which orthogonally diagonalize the matrix $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

4. (a) Use the Gram-Schmidt process to find an **orthonornal** basis for

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$$

   (b) Find the least-squares solution of $A\vec{x} = \vec{b}$ when $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

5. (a) Find the inverse of the matrix $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 2 & 3 \end{bmatrix}$.

   (b) Determine if each of the following sets of vectors is linearly independent or not. Justify each answer carefully:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}, \quad S_3 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

(c) The following matrices are row equivalent:

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 \\ -1 & -2 & -2 & -3 & 0 \\ 2 & 4 & 4 & 7 & -1 \\ 1 & 2 & 2 & 5 & -2 \\ 1 & 2 & 4 & 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the following:

i. A basis for Row(A)

ii. The dimension of Col(A)

iii. A basis for Nul(A)

6. (a) Recall that $\mathbb{P}_2(\mathbb{R})$ denotes the vector space of all polynomials of degree less than or equal to 2.

Given the linear transformation $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 + 3a_2 \\ 2a_1 \end{bmatrix}$$

Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2(\mathbb{R})$ and the standard basis for $\mathbb{R}^2$.

(b) Let $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\vec{b}_1\right) = 2\vec{b}_1 + \vec{b}_2 \quad \text{and} \quad T\left(\vec{b}_2\right) = -\vec{b}_1 + 3\vec{b}_2$$

i. Explain why $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2\right\}$ is a basis of $\mathbb{R}^2$.

ii. Find $[T]_{\mathcal{B}}$ (the matrix for T relative to the basis $\mathcal{B}$).

iii. Find M, the standard matrix of T (that is M such that $T\vec{x} = M\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$).