MATH/AMSC 673 - Fall 2011

Homework 1 - Due Septembre 19

1. Suppose that g is a C^1 function. Find an explicit formula for the solution of the initial value problem

$$\begin{cases} u_t + \mathbf{b} \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ are constants.

Hint: Convert this PDE into a transport equation by multiplying it by a suitable exponential.

2. Show that the minimal surface equation

$$\text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

can be written in two dimension as (denoting (x, y) the variable in $\mathbb{R}^2$)

$$u_{xx}(1 + u_y^2) + u_{yy}(1 + u_x^2) - 2u_x u_y u_{xy} = 0.$$

3. Suppose that $u = u(x, y) = v(r)$ (where $r = \sqrt{x^2 + y^2}$) is a radially symmetric solution of the minimal surface equation

$$u_{xx}(1 + u_y^2) + u_{yy}(1 + u_x^2) - 2u_x u_y u_{xy} = 0.$$

Show that v satisfies

$$v_{rr} + \frac{v_r}{r}(1 + v_r^2) = 0.$$

Use this result to show that $u(x, y) = \rho \log(r + \sqrt{r^2 - \rho^2})$ is a solution for any given $\rho > 0$ and for all $r = \sqrt{x^2 + y^2} > \rho$. (You can first compute $w = v_r$ and then check that w satisfies $(log(w))_r = -\frac{1}{r}(1 + w^2)$).

4. Prove that the Laplace equation $\Delta u = 0$ is rotation invariant. That is, if $\mathbf{R}$ is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(\mathbf{R}x), \quad x \in \mathbb{R}^n,$$

then $\Delta v = 0$.

5. We say that a function $v \in C^2(\overline{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Prove that if v is subharmonic then

$$v(x_0) \leq \fint_{B(x_0,r)} v(x)\, dx \quad \text{for all } B(x,r) \subset U.$$

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function (so $\phi''(s) \geq 0$ for all $s \in \mathbb{R}$). Assume that u is harmonic and set $v = \phi(u)$. Prove that v is subharmonic.