

MATH/AMSC 673 - Fall 2011  
Homework 2 - Due Friday Sept. 30

1. Suppose that  $u \in C^2(\mathbb{R}_+^2) \cap C^0(\overline{\mathbb{R}_+^2})$  is harmonic in  $\mathbb{R}_+^2$  and bounded in  $\overline{\mathbb{R}_+^2}$ . Prove the following maximum principle:

$$\sup_{\mathbb{R}_+^2} u = \sup_{\partial\mathbb{R}_+^2} u$$

(Hint: Consider the harmonic function  $u_\varepsilon(x, y) = u(x, y) - \varepsilon \log(x^2 + (y+1)^2)$  for  $\varepsilon > 0$ )

2. Use Poisson's formula for the ball to prove that if  $u$  is positive and harmonic in  $B(0, r)$ , then

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0)$$

(this is an explicit form of Harnack's inequality)

3. Let  $\mathcal{U}$  be a bounded open subset of  $\mathbb{R}^n$  with  $C^1$  boundary and let  $f \in C^0(\mathcal{U})$ . Consider the energy functional

$$I[u] = \int_{\mathcal{U}} \sqrt{1 + |Du|^2} - fu \, dx$$

and the set  $\mathcal{A} = \{u \in C^2(\overline{\mathcal{U}}); u = g \text{ on } \partial\mathcal{U}\}$ . Show that if  $u \in \mathcal{A}$  is such that

$$I[u] = \min_{w \in \mathcal{A}} I[w]$$

then  $u$  solves the following equation

$$-\operatorname{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = f \quad \text{in } \mathcal{U}.$$

4. Let  $\mathcal{U}^+$  denote the open half ball

$$\mathcal{U}^+ = \{x \in \mathbb{R}^n; |x| < 1, x_n > 0\}.$$

Assume that  $u \in C^2(\mathcal{U}^+) \cap C^0(\overline{\mathcal{U}^+})$  satisfies

$$\Delta u = 0 \text{ in } \mathcal{U}^+$$

$$u = 0 \text{ on } \partial\mathcal{U}^+ \cap \{x_n = 0\}.$$

Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for  $x \in \mathcal{U} = B(0, 1)$ .

Prove that  $v \in C^2(\mathcal{U})$  and that  $v$  is harmonic in  $\mathcal{U}$ .

(Hint: Use Poisson's formula for the unit ball and Theorem 15 in Evans book)

5. Let  $\mathcal{U}$  be a connected bounded open subset of  $\mathbb{R}^n$  with  $C^1$  boundary. Consider the boundary value problem

$$\begin{cases} -\Delta u + K^2 u = 0 & \text{in } \mathcal{U} \\ u + \frac{\partial u}{\partial \nu} = \phi & \text{on } \partial\mathcal{U} \end{cases}$$

where  $\phi \in C^0(\partial\mathcal{U})$ .

- (a) Use an energy method to show that there exists at most one solution in  $C^2(\overline{\mathcal{U}})$  of this boundary value problem.
- (b) Show that if  $u \in C^2(\overline{\mathcal{U}})$  satisfies the nonhomogeneous problem

$$\begin{cases} -\Delta u + K^2 u = f & \text{in } \mathcal{U} \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{U} \end{cases}$$

then  $u$  minimizes the functional

$$I[w] = \int_{\mathcal{U}} \frac{1}{2} |Dw|^2 + \frac{1}{2} K^2 w^2 - f w \, dx + \int_{\partial\mathcal{U}} \frac{1}{2} w^2 \, dS$$

on the set  $\mathcal{A} = \{w \in C^2(\mathcal{U}) \mid w + \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\mathcal{U}\}$ .