MATH/AMSC 673 - Fall 2011

Homework 3 - Due Oct. 10

1. Let U be a bounded subset of $\mathbb{R}^n$, and let $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$ satisfy

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

   Show that there exists a constant C depending only on n and U such that

$$\max_{x \in U} |u(x)| \leq C(\max_{x \in U} |f(x)| + \max_{x \in \partial U} |g(x)|)$$

   Hint: Construct a function w satisfying $w \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$, $-\Delta w \geq \max_{x \in U} |f(x)|$ in U and $w \geq \max_{x \in \partial U} |g(x)|$ on ∂U), and use the comparison principle.

2. Assume that U is an open, bounded, connected subset of $\mathbb{R}^n$. Use (a) energy methods and (b) the maximum principle (together with Hopf's lemma) to show that if u is a smooth solution of the Neumann BVP

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

   then u is constant in U.

3. Suppose that u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.
   (a) Show that $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
   (b) Use (a) to show that $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

4. Find an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

   Hint: Consider the function $v(x,t) = u(x,t)e^{ct}$.

5. Let $u \in \mathcal{C}^2(U \times (0,\infty)) \cap \mathcal{C}(\overline{U} \times [0,\infty))$ be a solution of

$$\begin{cases} u_t - \Delta u = \sin(u) & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times (0,\infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

Show that if $g(x) \leq 1$, then $u(x,t) \leq e^t$ for all $x \in U$, $t > 0$.