MATH/AMSC 673 - Fall 2011

Homework 4 - Due Oct. 24

1. Solve the following Cauchy problem in $\mathbb{R}$ (without using the fundamental solution):

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

where $g(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$ and $g(0) = 1/2$.

Hint: Look for a solution in the form $u(x, t) = \phi(\frac{x}{\sqrt{t}})$.

2. Let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary ∂U, and let $f : \mathbb{R} \to \mathbb{R}$ be smooth function with $|f'| \leq K$. Use an energy method to show that there exists at most one smooth solution of

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } U \times (0, \infty) \\ u = h & \text{on } \partial U \times (0, \infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

3. (a) Find an explicit formula for the solution of the IBVP:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{on } \{0\} \times (0, \infty) \\ u(x, 0) = g(x) & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

where $g \in C(\overline{\mathbb{R}_+}) \cap L^\infty(\mathbb{R}_+)$ satisfies $g(0) = 0$.

Hint: Use the reflection method seen in class for the wave equation.

(b) Find an explicit formula for the solution of the IBVP:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u(0, t) = g(t) & \text{on } \{0\} \times (0, \infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

where g is a smooth function satisfying $g(0) = 0$.

Hint: Use the function $v(x, t) = u(x, t) - g(t)$.

4. Write an explicit formula for the solution of

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } |x| \le t \\ u(-t,t) = \alpha(t) & \text{for } t \ge 0 \\ u_t(t,t) = \beta(t) & \text{for } t \ge 0 \end{cases}$$

where α and β are smooth functions.

5. Write an explicit formula for the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u_t = a u_x & \text{on } \{x = 0\} \times (0,\infty) \end{cases}$$

when $a \neq -c$ and g, h are C^2 functions that vanish near $x = 0$.

Show that no solution exists in general if $a = -c$.