1) $u(x,t) = \phi\left(\frac{x}{\sqrt{t}}\right)$

$u_t = -\frac{x}{2t^{3/2}}\,\phi'\left(\frac{x}{\sqrt{t}}\right)$

$u_x = \frac{1}{\sqrt{t}}\,\phi'\left(\frac{x}{\sqrt{t}}\right) \qquad u_{xx} = \frac{1}{t}\,\phi''\left(\frac{x}{\sqrt{t}}\right)$

$-\frac{x}{2t^{1/2}}\,\phi'\left(\frac{x}{\sqrt{t}}\right) - \phi''\left(\frac{x}{\sqrt{t}}\right) = 0$

$+\frac{1}{2}y\,\phi'(y) + \phi''(y) = 0$

$\frac{\phi''}{\phi'} = -\frac{1}{2}y$

$\ln|\phi'| = -\frac{1}{4}y^2 \qquad \phi'(y) = e^{-y^2/4}\,C_0$

So $\phi(y) = C_0 \int_0^y e^{-y^2/4}\,dy + C_1$

So $\boxed{u(x,t) = C_0 \int_0^{x/\sqrt{t}} e^{-y^2/4}\,dy + C_1}$

Note that $u \in C^\infty(\mathbb{R}\times(0,+\infty))$ is a solution of $u_t - u_{xx} = 0$

Initial Condition: $u(x,t) \xrightarrow[x>0]{t\to 0} C_0 \int_0^{\infty} e^{-y^2/4}\,dy + C_1$

$u(x,t) \xrightarrow[x<0]{t\to 0} C_0 \int_0^{-\infty} e^{-y^2/4}\,dy + C_1 = -C_0 \int_0^{\infty} e^{-y^2/4}\,dy + C_1$

$u(0,t) = C_1$

We want $\boxed{C_1 = 1/2}$, $\boxed{C_0 \int_0^{\infty} e^{-y^2/4}\,dy = \frac{1}{2}}$

2) u_1, u_2 solutions

$$(u_1-u_2)_t - \Delta(u_1-u_2) = f(u_1)-f(u_2)$$

$$\int_U (u_1-u_2)_t\,(u_1-u_2) - \int_U \Delta(u_1-u_2)(u_1-u_2) = \int\big(f(u_1)-f(u_2)\big)(u_1-u_2)$$

$$\frac{d}{dt}\int \frac{(u_1-u_2)^2}{2}\,dx + \int_U |D(u_1-u_2)|^2\,dx = \int\big(f(u_1)-f(u_2)\big)(u_1-u_2)$$

where we used the fact that $u_1-u_2=0$ on ∂U.

We write $|f(u_1)-f(u_2)| \leq |f'(\xi)|\,|u_1-u_2| \leq K\,|u_1-u_2|$

So $$\frac{d}{dt}\int \frac{(u_1-u_2)^2}{2}\,dx \leq K\int (u_1-u_2)^2\,dx$$

let $Y(t) = \int \frac{(u_1-u_2)^2}{2}\,dx$ then $Y' \leq KY$

so $Y(t) \leq Y(0)\,e^{Kt}$

$Y(0) = \int \frac{(g-g)^2}{2}\,dx = 0$ so $Y(t)=0$ for all t.

$u_1 = u_2$ for all t in U.

3) (a) $\tilde{g}(x) = \begin{cases} g(x) & \text{if } x > 0 \\ -g(-x) & \text{if } x < 0 \end{cases}$

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } x > 0 \\ -\tilde{u}(-x,t) & \text{if } x < 0 \end{cases}$$

$\tilde{u}$ solut° of $\begin{cases} \tilde{u}_t - \tilde{u}_{xx} = 0 \\ \tilde{u}(x,0) = \tilde{g}(x) \end{cases}$

So $\tilde{u}(x,t) = \int_{\mathbb{R}} \phi(x-y,t)\, \tilde{g}(y)\, dy$ where $\phi(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{4t}}$

$$= \int_0^\infty \phi(x-y,t)\, g(y)\, dy - \underbrace{\int_{-\infty}^0 \phi(x-y,t)\, g(-y)\, dy}_{= + \int_0^\infty \phi(x+y,t)\, g(y)\, dy}$$

So $u(x,t) = \int_0^\infty \left[\phi(x-y,t) - \phi(x+y,t)\right] g(y)\, dy$ for $x \in \mathbb{R}_+$, $t > 0$

Remark: It is easy to check that u is $\mathcal{C}^\infty(\mathbb{R}_+ \times (0,+\infty))$ and solves $u_t - u_{xx} = 0$

Furthermore $u(0,t) = \int_0^\infty \underbrace{\left[\phi(-y,t) - \phi(y,t)\right]}_{=0} g(y)\, dy$

and $u = \tilde{u}(x,t) \xrightarrow[\substack{t \to 0 \\ x > 0}]{} \tilde{g}(x) = g(x)$ (as seen in class)

3.b) Let $v(x,t) = u(x,t) - g(t)$

then
$$\begin{cases} v_t - v_{xx} = -g'(t) \\ v(0,t) = 0 \\ v(x,0) = 0 \end{cases}$$

the odd reflection $\tilde{v}$ of v solves

$$\tilde{v}_t - \tilde{v}_{xx} = \begin{cases} -g'(t) & \text{on } \mathbb{R}_+ \times (0,\infty) \\ g'(t) & \text{on } \mathbb{R}_- \times (0,\infty) \end{cases}$$

(and satisfies $\tilde{v}(0,t) = 0$, so $\tilde{v} = v$ on $\mathbb{R}_+ \times (0,\infty)$)

So
$$\tilde{v}(x,t) = \int_0^t \int_0^\infty \Phi(x-y,t-s)\,(-g'(s))\,dy\,ds$$
$$+ \int_0^t \int_{-\infty}^0 \Phi(x-y,t-s)\,g'(s)\,dy\,ds$$
$$= \int_0^t \int_0^\infty \left[\Phi(x-y,t-s) - \Phi(x+y,t-s)\right](-g'(s))\,dy\,ds$$

and $u(x,t) = v(x,t) + g(t)$

So
$$u(x,t) = \int_0^t \int_0^\infty \left[\Phi(x+y,t-s) - \Phi(x-y,t-s)\right] g'(s)\,dy\,ds + g(t)$$

General solution of $u_{tt} - u_{xx} = 0$:

$$u(x,t) = F(x+t) + G(x-t)$$

$$u(-t,t) = F(0) + G(-2t) = \alpha(t) \qquad t \geq 0$$
$$u_t(t,t) = F'(2t) - G'(0) = \beta(t) \qquad t \geq 0$$

We can fix $F(0) = 0 \Rightarrow G(t) = \alpha(-t/2)$ for $t \leq 0$

$$G'(0) = -\frac{1}{2}\alpha'(0)$$

So $F'(2t) = \beta(t) - \frac{1}{2}\alpha'(0)$

$$F'(t) = \beta(t/2) - \frac{1}{2}\alpha'(0)$$

So $F(t) = \int_0^t \beta(s/2)\,ds - \frac{1}{2}\alpha'(0)\,t \qquad t \geq 0$

and $u(x,t) = \int_0^{x+t} \beta(s/2)\,ds - \frac{1}{2}\alpha'(0)(x+t) + \alpha\left(\frac{t-x}{2}\right)$

5) General solut°

$$u(x,t) = F(x+ct) + G(x-ct)$$

F, G such that

$$\begin{cases} F(x) + G(x) = g(x) \\ c\,F'(x) - c\,G'(x) = h(x) \end{cases} \qquad \text{for all } x > 0$$

and

$$c\,F'(ct) - c\,G'(-ct) = a\left(F'(ct) + G'(-ct)\right) \qquad \text{for all } t > 0$$

The first two cdt° determine F and G for $x > 0$:

$$\begin{cases} F(x) + G(x) = g(x) \\ F(x) - G(x) = \frac{1}{c}\int_0^x h(y)\,dy + c_0 \end{cases}$$

So

$$\boxed{\begin{array}{ll} F(x) = \frac{1}{2} g(x) + \frac{1}{2c}\int_0^x h(y)\,dy + \frac{c_0}{2} & x > 0 \\ G(x) = \frac{1}{2} g(x) - \frac{1}{2c}\int_0^x h(y)\,dy - \frac{c_0}{2} & x > 0 \end{array}}$$

(we can take $c_0 = 0$)

Last condition determines G for $x < 0$:

$$c\,F'(x) - c\,G'(-x) = a\left(F'(x) + G'(-x)\right) \qquad x > 0$$

So $$(a+c)\,G'(-x) = (c-a)\,F'(x) = (c-a)\left(\frac{1}{2} g'(x) + \frac{1}{2c} h(x)\right)$$

So $$G(x) = -\frac{c-a}{a+c}\left(\frac{1}{2} g(-x) - \frac{1}{2c}\int_0^{-x} h(y)\,dy\right) + c_1 \quad \text{for } x < 0$$

take c_1 such that G is continuous at 0. ($\underline{\underline{c_1 = 0}}$) then u is C^2

$\underline{a = -c}$ then the last condition requires $F'(x) = 0$ for all $x > 0$