MATH/AMSC 673 - Fall 2011

Homework 5 - Due Nov. 11

1. Show that there is at most one smooth solution to the equation

$$u_{tt} + cu_t - u_{xx} = f$$

in the domain $(0,1) \times (0,\infty)$ with initial conditions $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ and boundary conditions $u(0,t) = u(1,t) = 0$ (you can start by assuming $c \geq 0$, but the case $c < 0$ is required also).

2. Let u be a C^2 solution of $u_{tt} - \Delta u = 0$ in $\mathbb{R}^2 \times (0,\infty)$, with

$$u(x,0) = 0, \qquad u_t(x,0) = g(x), \quad x \in \mathbb{R}^2$$

where g is a smooth function satisfying $g(x) = 0$ for $|x| > a$.

   (a) Show that there exists a constant C such that $|u(x,t)| \leq \frac{C}{t}$ for $t \geq 2(|x| + a)$. **Hint:** Use Poisson formula

   (b) Show that $\lim_{t\to\infty} tu(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} g(y)\,dy$ for all $x \in \mathbb{R}^2$.

3. Use characteristics to solve the linear equation $x_1 u_{x_1} + x_2 u_{x_2} = 2u$ in $\{(x_1,x_2),;\, x_1 \in \mathbb{R},\ x_2 > 1\}$ with boundary condition $u(x_1,1) = g(x_1)$ (and sketch the projected characteristics $s \mapsto x(s)$).

4. Consider the equation

$$-(u_{x_1})^2 + (u_{x_2})^2 + x_2^2 = 0, \qquad x_1 \in \mathbb{R},\ x_2 > 0$$

with boundary condition $u(x_1,0) = g(x_1)$ where g is a C^1 function satisfying $g'(y) > 0$ for all y.

   (a) Find explicitly the characteristics $x_1(s)$, $x_2(s)$ starting from the point $(y,0)$ (assuming $u_{x_2}(x,0) \geq 0$).

   (b) In the case $g(x_1) = x_1$, sketch the characteristics and write down an explicit solution (your answer may depend on the function $I(s) = \int_0^s \sin^2(r)\,dr$).

5. Find a smooth solution of

$$u_t + (u_x)^4 = 0 \quad x \in \mathbb{R},\ t > 0$$

with initial condition $u(x,0) = \frac{3}{4}x^{4/3}$.