1) $v = u_1 - u_2$ sol. of

$$\begin{cases} v_{tt} + c\,v_t - v_{xx} = 0 & \text{in } (0,1)\times(0,\infty) \\ v = 0,\ v_t = 0 & \text{on } (0,1)\times\{0\}. \end{cases}$$

$$\int v_t v_{tt} + c\,v_t^2 - v_{xx} v_t = 0$$

$$\frac{d}{dt}\underbrace{\int \frac{v_t^2}{2} + \frac{v_x^2}{2}\,dx}_{E(t)} + c\int v_t^2\,dx = 0$$

if $c \geq 0$ then $E'(t) \leq 0$ so $E(t) \leq E(0) = 0$

Hence $v = 0$ in $(0,1)\times(0,\infty)$

if $c < 0$ then $E'(t) = -c\int v_t^2\,dx \leq 2|c|\,E(t)$

So $E(t) \leq E(0)\,e^{2|c|t} = 0$

Hence $E(t) = 0$ and so $v = 0$ in $(0,1)\times(0,\infty)$

2) Poisson's formula (with $u(x,0) = 0$)

$$u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{t^2 g(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy \qquad x \in \mathbb{R}^2,\ t > 0$$

$t > 2(|x| + a) \implies B(0,a) \subset B(x,t)$, so

$$u(x,t) = \frac{1}{2}\frac{1}{\pi t^2}\int_{B(0,a)} \frac{t^2 g(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy$$

So $$u(x,t) = \frac{1}{2\pi} \int_{B(0,a)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy$$

If $y \in B(0,a)$ then $|y-x| \leq |x| + a$

So $t \geq 2(|x|+a) \Rightarrow \frac{t}{2} \geq |y-a|$

and we deduce

$$t^2 - |y-x|^2 \geq t^2 - \frac{t^2}{4} = \frac{3t^2}{4}$$

It follows $$|u(x,t)| \leq \frac{1}{2\pi} \frac{2}{\sqrt{3}} \int_{B(0,a)} \frac{|g(y)|}{t}\, dy$$

$$\leq \frac{C}{t} \quad \text{with} \quad C = \frac{1}{\sqrt{3}\pi} \int_{B(0,a)} |g(y)|\, dy$$

b) $$t\, u(x,t) = \frac{1}{2\pi} \int_{B(0,a)} \frac{t}{(t^2 - |y-x|^2)^{1/2}}\, g(y)\, dy \qquad \frac{\text{fix } x \in \mathbb{R}^2}{t > 2(|x|+a)}$$

We have

- $\dfrac{t}{(t^2 - |y-x|^2)^{1/2}} \xrightarrow[t \to +\infty]{} 1$ for all $y \in B(0,a)$
- $\dfrac{t}{(t^2 - |y-x|^2)^{1/2}} \leq C$ if $t > 2(|x|+a)$

Lebesgue dominated convergence theorem implies

$$\int \frac{t}{(t^2 - |y-x|^2)^{1/2}}\, g(y)\, dy \xrightarrow[t \to +\infty]{} \int_{B(0,a)} g(y)\, dy = \int_{\mathbb{R}^2} g(y)\, dy$$

Hence the result.

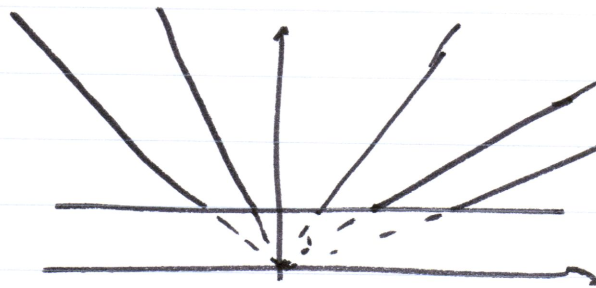
3] a) C.E.: $\begin{cases} x_1'(s) = x_1(s) \\ x_2'(s) = x_2(s) \\ z'(s) = 2z(s) \end{cases}$ $\quad x_1(0) = x_0$, $\quad x_2(0) = 1$, $\quad z(0) = u(x_0, 1) = g(x_0)$

So:

$x_1(s) = x_0 e^{s}$

$x_2(s) = e^{s}$

$z(s) = z(0) e^{2s} = g(x_0) e^{2s}$



Fix $(x_1, x_2) \in \mathbb{R}^2$, $x_2 > 0$ then

$s = \ln x_2$ and $x_0 = \dfrac{x_1}{e^s} = \dfrac{x_1}{x_2}$

So $u(x_1, x_2) = g\left(\dfrac{x_1}{x_2}\right) x_2^2$

4) $F(p, u, x) = -p_1^2 + p_2^2 + x_2^2$

$D_x F = \begin{pmatrix} 0 \\ 2x_2 \end{pmatrix} \qquad D_p F = \begin{pmatrix} -2p_1 \\ 2p_2 \end{pmatrix} \qquad F_z = 0$

$$\begin{cases} p_1'(s) = 0 \\ p_2'(s) = -2x_2(s) \\ z'(s) = -2p_1^2 + 2p_2^2 \\ x_1'(s) = -2p_1 \\ x_2'(s) = 2p_2 \end{cases}$$

$p' = -D_x F - F_z p$

$z' = D_p F \cdot p$

$x' = D_p F$

So $p_1(s) = p_1^0$, $x_1(s) = -2p_1^0 s + y$

$$\begin{cases} p_2' = -2x_2 \\ x_2' = 2p_2 \\ x_2(0) = 0 \end{cases} \Rightarrow \begin{array}{l} p_2 = p_2^0 \cos(2s) \\ x_2 = p_2^0 \sin(2s) \end{array}$$

Finding p_1^0, p_2^0: $\quad u(x_1, 0) = g(x_1)$

$\Rightarrow u_{x_1}(y, 0) = g'(y)$ so $p_1^0 = g'(y)$

$F(p, u, x) = 0 \Rightarrow p_2^{0\,2} = p_1^{0\,2} \to p_2^0 = \pm g'(y) = g'(y)$

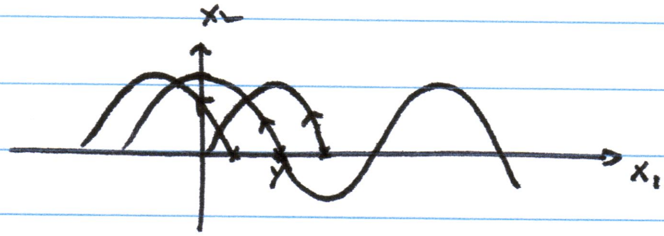
if $u_{x_2}(x,0) > 0$

We then $\begin{cases} x_1(s) = y - 2g'(y)\, s \\ x_2(s) = g'(y) \sin(2s) \end{cases}$

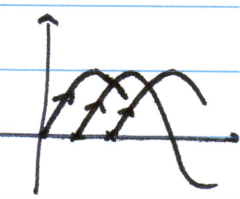
b) $g'(x_1) = 1$ so $\begin{cases} x_1(s) = y - 2g'(y)s = y - 2s \\ x_2(s) = g'(y)\sin(2s) = \sin(2s) \end{cases}$

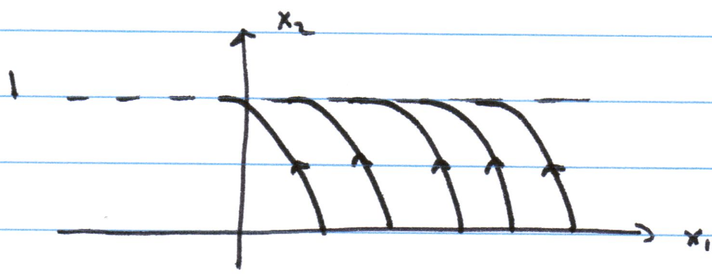
we have $2s = y - x_1$, so $x_2 = \sin(y - x_1)$

$= -\sin(x_1 - y)$



Remark: with $p_1^0 = -g'(y) = -1$, we would get





We can define a sol by taking only $2s \in [0, \pi/2]$

x_1, x_2 in $\mathbb{R} \times [0,1] \Rightarrow \begin{cases} x_1 = y - 2s \\ x_2 = \sin 2s \end{cases}$ $\quad y = x_1 + \arcsin x_2$, $\quad s = \frac{1}{2} \arcsin x_2$

then $z'(s) = -2\, {p_1^0}^2 + 2\, {p_2^0}^2 \cos^2(2s)$

$= -2(g'(y))^2 + 2(g'(y))^2 \cos^2(2s)$

$= -2 + 2\cos^2(2s) = -2\sin^2(2s)$

So $z(s) = -2\int_0^s \sin^2(2r)\,dr + \underbrace{z(0)}_{= g(y)}$

$= -\int_0^{2s} \sin^2(r)\,dr + y = -I(2s) + y$

So $u(x_1, x_2) = -I(\arcsin x_2) + x_1 + \arcsin x_2 \qquad x_1 \in \mathbb{R},\ x_2 \in [0,1]$

5] $Y = (x,t)$

$q = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ $\qquad G(q, \cancel{z}, y) = p_2 + p_1^4$

$$D_q G = \begin{pmatrix} 4p_1^3 \\ 1 \end{pmatrix} \qquad D_z G = 0 \qquad D_x G = 0$$

So
$$\begin{cases} p_1' = 0 \\ p_2' = 0 \\ z'(s) = 4p_1^4 + p_2 & z(0) = \frac{3}{4}(x^0)^{4/3} \\ x_1'(s) = 4p_1^3 & x_1(0) = x^0 \\ x_2'(s) = 1 \quad \leftrightarrow \quad t = s \end{cases}$$

So $p_1 = p_1^0$, $p_2 = p_2^0$

$u(x,0) = \frac{3}{4} x^{4/3}$ so $u_x(x,0) = x^{1/3}$ Hence $\underline{p_1^0 = (x^0)^{1/3}}$

and $p_2^0 + p_1^{0\,4} = 0$ so $p_2^0 = -p_1^{0\,4} = \underline{-(x^0)^{4/3}}$

Finally $x'(s) = 4p_1^{0\,3} = 4x^0 \implies \underline{x(s) = 4x^0 s + x^0}$

and $z'(s) = 4p_1^{0\,4} + p_2^0 = 4x^{0\,4/3} - (x^0)^{4/3} = 3x^{0\,4/3}$

$$\implies z(s) = 3x^{0\,4/3} s + \frac{3}{4}(x^0)^{4/3}$$

Let $(x,t) \in \mathbb{R} \times (0,\infty)$ $\quad s = t$, $\quad x = 4x^0 t + x^0 = (4t+1)x^0$

So $x^0 = \dfrac{x}{4t+1}$

Hence $u(x,t) = \left(\dfrac{x}{4t+1}\right)^{4/3}\left(3t + \dfrac{3}{4}\right) = \dfrac{3}{4}\dfrac{x^{4/3}}{(4t+1)^{1/3}}$