MATH/AMSC 673 - Fall 2011

Homework 6 - Due Nov. 23

1. Using the method of **characteristics**, find an explicit solution of

$$u_t + \frac{1}{2}(u_x^2 + x^2) = 0 \qquad \text{on } \mathbb{R} \times (0, \infty)$$

with initial condition $u(x,0) = \frac{1}{2}x^2$.

2. (a) Let $H(p) = \frac{1}{r}|p|^r$ for $1 < r < \infty$. Compute the Legendre transform H^* of H.

   (b) Let $H(p) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}p_ip_j + \sum_{i=1}^n b_ip_i$ where $A = (a_{ij})$ is a symmetric positive definite matrix and $b \in \mathbb{R}^n$. Compute H^*.

3. Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be convex. We say that v belongs to the *subdifferential* of H at p if

$$H(q) \geq H(p) + v \cdot (q - p) \quad \forall q \in \mathbb{R}^n.$$

We write $v \in \partial H(p)$ (note that if H is differentiable at p, then $\partial H(p) = \{D_pH(p)\}$).

Prove

$$v \in \partial H(p) \iff p \in \partial L(v) \iff p \cdot v = H(p) + L(v)$$

where $L = H^*$.

4. Let u^1, u^2 be two solutions (given by Hopf-Lax formula) of

$$u_t + H(Du) = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

with initial condition $u^i = g^i$. Prove that

$$\sup_{x \in \mathbb{R}^n} |u^1(x,t) - u^2(x,t)| \leq \sup_{x \in \mathbb{R}^n} |g^1(x) - g^2(x)|, \qquad \forall t \geq 0.$$

5. Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E \end{cases} & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

it would give the solution

$$u(x,t) = \frac{1}{4t}\operatorname{dist}(x,E)^2$$

(where $\operatorname{dist}(x,E) = \inf\{|x-y|; y \in E\}$ denotes the distant of x to E).