Hw 6

1) $H(p,x) = \frac{1}{2}(p^2+x^2)$

$$\dot{x}(s) = D_p H(p,x) = p(s)$$
$$\dot{p}(s) = -D_x H(p,x) = -x(s)$$
$$\dot{z}(s) = D_p H \cdot p - H = \frac{1}{2}\left(p(s)^2 - x(s)^2\right)$$

Initial Conditions $x(0) = x^0$

$$p(0) = p^0 = Dg(x^0) = x^0$$
$$z(0) = z^0 = g(x^0) = \frac{1}{2}x^{0\,2}$$

$$\left.\begin{array}{l}\ddot{x} = -x \\ \ddot{p} = -p\end{array}\right\} \Rightarrow \begin{cases} x(s) = x^0(\cos s + \sin s) \\ p(s) = x^0(-\sin s + \cos s)\end{cases}$$

and $\dot{z}(s) = \frac{1}{2}p^2 - \frac{1}{2}x^2$

$$= -2x^{0\,2}(\cos s)(\sin s)$$

So $z(s) = \int_0^s -2x^{0\,2}(\cos t)(\sin t)\,dt + \frac{x^{0\,2}}{2}$

$$= \frac{1}{2}x^{0\,2} - x^{0\,2}\sin^2 t$$

(x,t) in $\mathbb{R} \times (0,\infty) \Rightarrow$

- $s = t$
- $x^0 = \dfrac{x}{\sin t + \cos t}$ $\quad t < \frac{3\pi}{4}$

So $u(x,t) = z(t) = \dfrac{x^2}{(\sin t+\cos t)^2}\left(\frac{1}{2} - \sin^2 t\right)$ $\quad t < \frac{3\pi}{4}$

2 a) $\quad H(p) = \frac{1}{r}|p|^r$

$$L(v) = H^*(v) = \max_{p \in \mathbb{R}^n} \{ \underbrace{p \cdot v - H(p)}_{f_v(p)} \}$$

critical point of f_v : $\quad p = 0$ and $\quad Df_v(p) = v - |p|^{r-2} p = 0$

$$\text{so} \quad v = |p|^{r-2} p$$

$$\text{or} \quad |p| = |v|^{1/r-1}$$

$$p \cdot v = |p|^r = |v|^{r/r-1}$$

so $\quad f_v(0) = 0 \quad$ or $\quad f_v(\bar{p}) = |v|^{\frac{r}{r-1}} - \frac{1}{r}|v|^{r/r-1}$

$$= \left(\frac{r-1}{r}\right)|v|^{r/r-1} > 0$$

Hence $\boxed{L(v) = \frac{r-1}{r} |v|^{\frac{r}{r-1}}}$

b) $\quad H(p) = \frac{1}{2} p^T A p + b \cdot p \qquad$ differentiable, uniformly convex

$$\cancel{L(v) = \max_{p \in \mathbb{R}^n} \{ \underbrace{p \cdot v - \tfrac{1}{2} p^T A p - b \cdot p}_{f_v(p)} \}}$$

$$L(v) = p \cdot v - H(p) \iff v = D_p H(p)$$

$v = D_p H(p) = A p + b \qquad$ (A symmetric)

$$\text{so} \quad p = A^{-1}(b - v)$$

and $\quad p \cdot v = p^T A p + b \cdot p$

Hence $\quad L(v) = \frac{1}{2} p^T A p = \frac{1}{2} (b - v)^T A^{-1} A A^{-1} (b - v)$

$$\boxed{L(v) = \frac{1}{2} (b - v)^T A^{-1} (b - v)}$$

3] <u>If $v \in \partial H(p)$</u>, then

$$v \cdot p - H(p) \geq v \cdot q - H(q) \quad \text{for all } q \in \mathbb{R}^n$$

So $\quad v \cdot p - H(p) \geq \max_q \{v \cdot q - H(q)\} = L(v)$

Hence $\quad L(v) \geq v \cdot p - H(p) \geq L(v)$

and so $\quad \underline{L(v) = v \cdot p - H(p)}$

Conversely, if $L(v) = v \cdot p - H(p)$, then

$$\max_q \{v \cdot q - H(q)\} = v \cdot p - H(p)$$

and so $\quad v \cdot q - H(q) \leq v \cdot p - H(p) \quad \forall q \in \mathbb{R}^n$

We deduce $\quad H(q) \geq H(p) + v \cdot (q - p) \quad \forall q \in \mathbb{R}^n$

Hence $\quad \underline{v \in \partial H(p)}$

We showed $\quad v \in \partial H(p) \iff L(v) + H(p) = p \cdot v$

To conclude, we can use the fact that $L^* = H$

4] $u^1 = \min \{ t L(\frac{x-y}{t}) + g^1(y) \}$ fix x & t

$= t L(\frac{x-y_1}{t}) + g^1(y_1)$ for some y_1

$u^2 \leq t L(\frac{x-y_1}{t}) + g^2(y_1)$ for the same y_1

so $u^2(x,t) - u^1(x,t) \leq g^2(y_1) - g^1(y_1) \leq \sup_{y \in \mathbb{R}^n} |g^2(y) - g^1(y)|$

Similarly (exchange the role of u^1 & u^2) we have

$u^1(x,t) - u^2(x,t) \leq \sup |g^2 - g^1|$

so $|u^1(x,t) - u^2(x,t)| \leq \sup_y |g^2(y) - g^1(y)|$ for all x, t.

5] $H(p) = |p|^2$ $D_p H(p) = 2p$ so $L(v) = \frac{1}{4}|p|^2$

So $u(x,t) = \min_{y \in \mathbb{R}^n} \{ \underbrace{\frac{t}{4} \frac{|x-y|^2}{t^2} + g(y)}_{= +\infty \text{ if } y \notin E} \}$

$= \min_{y \in E} \{ \frac{1}{4t} |x-y|^2 \}$

$= \frac{1}{4t} \left(\min_{y \in E} |x-y| \right)^2$

$$\boxed{u(x,t) = \frac{1}{4t} \operatorname{dist}(x, E)^2}$$