MATH/AMSC 673 - Fall 2011

Homework 7 - Due Friday December 9th

1. Consider the initial value problem

$$\begin{cases} u_t + uu_x + u = 0\,, & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = a\sin x & \text{on } \mathbb{R}\times\{0\} \end{cases}$$

   (a) Find the characteristic curves explicitly.

   (b) Show that if $a > 1$, then there does not exist a smooth solution defined for all time $t > 0$. Find the maximal time of existence of the smooth solution.

2. Compute explicitly the unique entropy solution of

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0\,, & \text{in } \mathbb{R}\times(0,\infty) \\ u = g & \text{on } \mathbb{R}\times\{0\} \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Draw a picture illustrating your answer.

3. Find an integral solution of

$$\begin{cases} u_t + (F(u))_x = 0\,, & \text{in } \mathbb{R}\times(0,\infty) \\ u = g & \text{on } \mathbb{R}\times\{0\} \end{cases}$$

when $F(u) = u^2 + u$ and $g(x) = \begin{cases} 1 & x < 0 \\ -3 & x > 0 \end{cases}$.

4. Find an entropy solution of

$$\begin{cases} u_t + u^2 u_x = 0\,, & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

with $g(x) = \begin{cases} 0 & x < 0 \\ -2 & x > 0 \end{cases}$.

Make sure that your solution satisfies the entropy condition.

Hint: The solution should involve a rarefaction wave of the form $v(x/t)$.

5. Suppose $u(x,t)$ is a smooth solution of $u_t + uu_x = 0$ for $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = g(x)$. Assume that g is a C^1 function such that $g(x) = 0$ if $x < -1$ and $g(x) = 1$ if $x > 1$ and $g'(x) > 0$ if $|x| < 1$. Show that for $t > 0$,

$$\lim_{r\to\infty} u(rx, rt) = \begin{cases} 0 & \text{if } x < 0 \\ x/t & \text{if } 0 < x < t \\ 1 & \text{if } x > t. \end{cases}$$