Self polarization and traveling wave in a model for cell crawling migration

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Abstract

In this paper, we prove the existence of traveling wave solutions for an incompressible Darcy's free boundary problem recently introduced in [6] to describe cell motility. This free boundary problem involves a nonlinear destabilizing term in the boundary condition which describes the active character of the cell cytoskeleton. By using two different methods, a constructive method via a graph analysis and a local bifurcation method, we prove that traveling wave solutions exist when the destabilizing term is strong enough.

1 Introduction

In this paper we study the existence of traveling wave solutions for the following two-dimensional free boundary problem, which models the dynamics of a living cell:

$$\begin{cases} \Delta P = 0 & \text{in } \Omega(t), \\ P + \beta f(V) = \gamma \kappa(t) & \text{on } \partial\Omega(t), \\ V = -\nabla P \cdot \mathbf{n} & \text{on } \partial\Omega(t), \\ \Omega(t=0) = \Omega_0. \end{cases} \tag{1.1}$$

The set $\Omega(t) \subset \mathbb{R}^2$ describes the domain occupied by a cell (at time t) whose boundary $\partial\Omega(t)$ moves with normal velocity V. This velocity is determined together with the unknown function P, which represents the pressure inside the cell. In (1.1), Ω_0 is a bounded domain of $\mathbb{R}^2$ (the initial configuration of the cell), κ is the mean curvature (positive for a circle) of the evolving free-boundary $\partial\Omega(t)$, $\mathbf{n}$ is the outwards-pointing unit normal on $\partial\Omega(t)$ and $\gamma, \beta > 0$ are some parameters. The given non-linearity

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$f : \mathbb{R} \to \mathbb{R}$ plays a crucial role in the paper and satisfies assumptions (A1)-(A4) listed below.

This model was derived in a recent work by the authors [6] as a sharp interface limit of a phase-field model to describe the motion of a cell on a 2D substrate. It can also be seen as the first order perturbation of a coupled free boundary model that shares similarities with the model introduced in [13] (see Section 2).

Throughout the paper, we assume that f satisfies the following assumptions:

(A1) f is $\mathcal{C}^2(\mathbb{R})$, monotone increasing and odd (so $f(0) = 0$),
(A2) $\lim_{x\to+\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$,
(A3) $f'(0) > 0$, $f''(0) = 0$,
(A4) $f''(x) \le 0\ \forall x > 0$ and $f''(x) \ge 0\ \forall x < 0$.

These assumptions are satisfied in particular by the nonlinearity $f(V) = -F_\tau(V)$ that we derived in [6] from the phase-field model (2.1) (for the double-well potential $W(\rho) = \rho^2(1-\rho^2)$). A prototype example of a function satisfying these assumptions is $f(x) = \tanh x$.

The originality of the problem (1.1) is in the boundary condition $(1.1)_2$ which describes the effects of polymerization. Indeed, unlike in the classical Hele-Shaw equation with surface tension (which corresponds to $\beta = 0$) the perimeter $\mathcal{P}(\Omega(t))$, defined by $\mathcal{P}(\Omega(t)) = \int_{\partial\Omega(t)} \mathrm{d}\sigma$ is not a Lyapunov functional for (1.1). Using a classical computation [17], we see that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(\Omega(t)) &= \int_{\partial\Omega(t)} \kappa V \,\mathrm{d}\sigma = -\frac{1}{\gamma}\int_{\partial\Omega(t)} P\nabla P\cdot n \,\mathrm{d}\sigma + \frac{\beta}{\gamma}\int_{\partial\Omega(t)} V f(V)\,\mathrm{d}\sigma \\ &= -\frac{1}{\gamma}\int_{\Omega(t)} |\nabla P|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{\beta}{\gamma}\int_{\partial\Omega(t)} V f(V)\ \mathrm{d}\sigma, \end{aligned} \tag{1.2}$$

where $\mathrm{d}\sigma$ denotes the infinitesimal length element of $\partial\Omega(t)$. Assumption (A1) implies

$$V f(V) \ge 0,$$

and so the two terms in the right-hand side of (1.2) have opposite effects. We thus see that the quantity $\beta f(V)$ in the boundary condition $(1.1)_2$ has a destabilzing effect. It models a force exerted on the membrane of the cell by the cytoskeleton which is responsible for the appearance of protrusions which in turn make the displacement of the cell possible (see [19, 12] for biological details).

On the other hand, we note that the model (1.1) is still area preserving:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Omega(t)| = \int_{\partial\Omega(t)} V \,\mathrm{d}\sigma = -\int_{\partial\Omega(t)} \nabla P\cdot\mathbf{n}\,\mathrm{d}\sigma = -\int_{\Omega(t)} \Delta P \,\mathrm{d}x\,\mathrm{d}y = 0\,, \tag{1.3}$$

and since $f(0) = 0$, it admits the disk as unique stationary solution (rest state) with constant pressure

$$P^* = \frac{\gamma}{R_0}, \tag{1.4}$$

where $R_0 > 0$ is such that $|B(0, R_0)| = |\Omega_0|$.

A remarkable feature of cell motility is the occurrence of sustained motion in a given direction without exterior impulse. This phenomenon, known as self-polarization [7], is mathematically described by the existence of traveling wave solutions and is the main topic of this paper.

Before stating our results about the existence of traveling wave solutions of (1.1), let us briefly comment on the literature. Moving interface problems have raised many interesting and challenging mathematical issues. A well known example is the Stefan problem which describes the dynamics of the boundary between ice and water. In the biophysical community, we find a large number of free boundary models to describe tumor and tissue growth, cell motility and other phenomena. We refer to [19, 14] for a review. Most of them are formulated through a fluid approach with surface tension. Some tumor growth models (e.g. [8, 9, 10]) resemble our model (1.1). However, there is an important difference: tumor growth naturally involves expanding domain while we consider here incompressible solutions. In the context of the motility of eukaryotic cells on substrates, various free boundary problems have been derived and studied, see [1, 2, 3, 15, 4, 16]. The models presented in [1, 2, 3, 15, 16] present some similarities with our model, but they are obtained as the limit of a second order equation of Allen-Cahn type while we obtain ours as the limit of a fourth order equation of Cahn-Hilliard type. The model recently proposed in [4] involves a coupled Hele-Shaw/ Keller-Segel type free boundary problem. The existence of traveling wave solutions for these models is proved in [1, 3, 4]. We note finally that traveling wave solutions for a moving boundary problem of Hele-Shaw type have been studied in the case with kinetic undercooling regularization [11].

Traveling wave solutions of (1.1) correspond to a fixed shape domain moving by translation with constant velocity in a given direction, that is

$$\Omega(t) = \Omega_0 + ct\mathbf{u}, \tag{1.5}$$

for some speed c and direction of motion $\mathbf{u}$.

It is the goal of this paper to prove the existence of non-trivial traveling wave solutions of (1.1) and thus validate the interest of this model to describe cell motility. In the sequel we refer to (Ω_0, c) as a traveling wave solution of (1.1) if the set $\Omega(t)$ defined by (1.5) is a solution of (1.1). The stationary solution B_{R_0} is a traveling wave solution with zero speed. It is referred to as trivial solution of (1.1).

Note that the problem is isotropic so we will always consider $\mathbf{u} = \mathbf{e}_x = (1, 0)$ and $c > 0$. In that case, the boundary velocity of $\Omega(t)$ given by (1.5) satisfies $V = c\,\mathbf{e}_x \cdot \mathbf{n}$, and a traveling-wave solution to (1.1) is defined as follows:

Definition 1.1. *A traveling wave solution to* (1.1) *is given by a domain* $\Omega_0 \in \mathbb{R}^2$*,* *a positive real number c and a function $P(\cdot)$ defined on Ω_0 satisfying*

$$\begin{cases} -\Delta P = 0 & \textit{in } \Omega_0, \\ P = \gamma\kappa - \beta f(c\,\mathbf{e}_x \cdot \mathbf{n}) & \textit{on } \partial\Omega_0, \\ -\nabla P \cdot \mathbf{n} = c\,\mathbf{e}_x \cdot \mathbf{n} & \textit{on } \partial\Omega_0. \end{cases} \tag{1.6}$$

Remarkably, condition (1.6) imposes that the entire fluid bulk flows at a uniform speed, that is $\nabla P = -(c, 0)$ in Ω_0. Indeed, the function $P{+}cx$ solves

$$\begin{cases} -\Delta P = 0 & \text{in } \Omega_0, \\ -\nabla[P + cx] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_0 \end{cases}$$

and thus satisfies $P{+}cx = \lambda$ in Ω_0. We deduce the following characterization of traveling wave solutions:

Proposition 1.2. *Any traveling wave solution to (1.1) moving with velocity $c > 0$ in the x-direction is given by a domain $\Omega_0 \subset \mathbb{R}^2$ and a real number λ such that the following condition holds*

$$\lambda - cx = \gamma\kappa - \beta f(c\mathbf{e}_x \cdot \mathbf{n}) \quad \textit{on } \partial\Omega_0. \tag{1.7}$$

Remark 1.3. *In this problem, the set Ω_0 and the speed c must be found together. The parameter λ can be seen as a Lagrange multiplier for the volume of Ω_0, but the problem is invariant by translation and any translation of Ω_0 leads to a solution of (1.7), with the same c but a different value of λ.*

Our first result is the following:

Theorem 1.4. *Assume that f satisfies assumptions (A1) – (A4). For all γ, $\lambda > 0$, there exists a one parameter family of traveling wave solutions of (1.1) (with $\mathbf{u} = \mathbf{e}_x$) (Ω_β, c_β) parametrized by $\beta \in [\frac{\gamma}{\lambda f'(0)}, \infty)$ and satisfying:*

(i) $c_\beta > 0$ when $\beta > \beta^* := \frac{\gamma}{\lambda f'(0)}$ and $c_\beta \to c_{\beta^*} = 0$ when $\beta \to \beta^*$.

(ii) The set Ω_β is a convex set with $C^{2,1}$ boundary of the form

$$\Omega_\beta = \{(x, y)\,;\, x_L < x < x_R,\ -h(x) < y < h(x)\}$$

with $x_L < 0 < x_R$ and for some C^3 function h satisfying $h'(0) = 0$.

(iii) At the point $m = (0, h(0)) \in \partial\Omega_\beta$ the normal vector $\mathbf{n}$ is the vertical vector $(0, 1)$ and the curvature is $\kappa(m) = \lambda/\gamma$. Furthermore, Ω_{β^*} is a disk of radius $R_c = \gamma/\lambda$.

Property (i) guarantees in particular that we are constructing non-trivial traveling wave solutions (i.e. not stationary solutions) when $\beta > \beta^*$. Property (ii) fixes the natural invariance by translation of the model. Property (iii) relates the value of the parameter λ to some geometric property of Ω_λ. It proves that each value of λ

yields of different family of traveling waves and it suggests that increasing values of λ correspond to sets with decreasing volume (something that we can check numerically but are not presently able to prove).

While the proof of this theorem is constructive, it does not clearly identify what happens for a cell of fixed volume when the value of β increases. Our next theorem will make this more precise: Using a bifurcation argument, we will prove that, for a fixed volume, a branch of traveling wave solutions with non zero speed emerges from the trivial solution B_{R_0} at $\beta = \frac{R_0}{f'(0)}$.

Theorem 1.5. *Let f satisfy assumptions (A1) – (A4) and assume furthermore that $f \in C^3$ and $f'''(0) < 0$. The problem (1.1) has a branch of traveling wave solutions with volume $|B_{R_0}|$ (in the x-direction) bifurcating from the radial solution B_{R_0} at $\beta = \frac{R_0}{f'(0)}$. The bifurcation is a pitchfork bifurcation, with:*

$$\begin{cases} \beta(s) &= \frac{R_0}{f'(0)} + \alpha s^2 + o(s^2), \\ c(s) &= s + o(s), \end{cases} \tag{1.8}$$

where $\alpha > 0$ and the parameter s takes value in an interval $(-\delta, \delta)$. For each s, the corresponding traveling wave solution solves (1.7) with $\lambda = \lambda(s) = \frac{\gamma}{R_0} + o(s^2)$.

This theorem shows that when $\beta \in (\frac{R_0}{f'(0)}, \frac{R_0}{f'(0)} + \eta)$ (for some $\eta > 0$), there exist traveling wave solutions with volume $|B_{R_0}|$ moving with positive speed (in any direction).

We note that our two theorems construct different families of traveling waves, since the first result fixes the Lagrange multiplier λ while the second one fixes the volume. But if we take fix $\lambda > 0$ in Theorem 1.4 and choose $R_0 = \gamma/\lambda$ in Theorem 1.5, then both theorems prove the existence of non trivial traveling wave solutions for $\beta > \beta^* = \frac{R_0}{f'(0)} = \frac{\gamma}{\lambda f'(0)}$, which converge to B_{R_0} when $\beta \to \beta^*$.

The rest of the paper is organized as follows. In Section 2 we briefly recall the biological justification of problem (1.1). In Section 3 we give the proof of Theorem 1.4 and in Section 4 we prove Theorem 1.5. Finally we give some conclusions in Section 5. Several technical computations are presented in appendix.

2 Biological justification of (1.1)

In this part, we briefly describe the biological origins of the problem (1.1). On the one hand, it is obtained as a sharp interface limit of a Cahn-Hilliard model describing the motion of a cell on a 2D substrate, see [6]. On the other hand it is the first order perturbation of a coupled free boundary model similar to the model introduced in [13].

2.1 Sharp interface limit of the phase field model introduced in [6]

In [6], we introduced the following phase-field model to describe the motion of a cell on a 2D substrate:

$$\begin{cases} \partial_t \rho = \operatorname{div}\left(\rho \nabla \left[\gamma\left(-\varepsilon \Delta \rho + \frac{1}{\varepsilon} W'(\rho)\right) + \phi\right]\right), \\ \partial_t \phi - \varepsilon \Delta \phi = \frac{1}{\varepsilon}\left(\beta \rho - \phi\right), \end{cases} \tag{2.1}$$

with $\varepsilon > 0$, $\gamma > 0$, $\beta \geq 0$ and W a double-well potential satisfying

$$W(0) = W(1) = 0, \quad W(\rho) > 0 \text{ if } \rho \neq 0,\, 1 \tag{2.2}$$

(for instance $W(\rho) = \rho^2(1-\rho)^2$).

From a modeling point of view, the system (2.1) is very simple. Two quantities are used to describe the cell: the phase field (or order parameter) ρ, describing everything that lies inside the cell (cytoskeleton, solvent, molecular motors...), and the myosin II, a molecular motor that assembles in minifilaments, interacts with actin, behaves as active crosslinkers and generates contractile or dilative stresses in the cytoskeleton network, whose concentration is denoted by ϕ. The main assumptions that lead to (2.1) are the following: (i) the cell velocity, v is given by the local actin flow, (ii) myosin II in the bulk is slowly diffusing, (iii) actin filaments undergo uniform bulk polymerization and depolymerization, (iv) the osmotic pressure involved in the network stress acts to saturate the linear instability causing gel phase separation and to smooth the interface between cytosol-rich and cytosol-poor regions. The underlying processes are: friction of the cytosol on the substrate together with the active character of the myosin II.

When $\varepsilon \ll 1$, we showed in [6] that this model is close to the free boundary problem (1.1) in which the cell is described by a set $\Omega(t)$ (so $\rho^\varepsilon(x,t) \sim \chi_{\Omega(t)}(x)$).

2.2 First order perturbation of a coupled free boundary model

Consider the cytoplasm as a confined viscous droplet that is driven by an active force induced by the cytoskeleton on its boundary (the cell membrane). Biologically, such a force can be generated either by polymerization of actin against the membrane or by contraction of cortical actomyosin filaments, which adhere to the membrane. We suppose that the active force is controlled by a diffusive chemical solute which is advected by the internal cytoplasmic flow. More precisely, we consider the following 2D free-boundary problem

$$\Delta P = 0 \qquad \text{in } \Omega(t)\,, \tag{2.3}$$

$$P + \beta f(-\nabla \phi \cdot n) = \gamma \kappa \qquad \text{on } \partial\Omega(t)\,, \tag{2.4}$$

$$V = -\nabla P \cdot n \qquad \text{on } \partial\Omega(t)\,. \tag{2.5}$$

where ϕ solves an advection-diffusion equation

$$\partial_t \phi + (a-1)\nabla P \cdot \nabla \phi - \Delta \phi = 0 \qquad \text{in } \Omega(t), \tag{2.6}$$

$$\nabla \phi \cdot n = a\phi \nabla P \cdot n \qquad \text{on } \partial\Omega(t), \tag{2.7}$$

where $a \in [0, 1)$ is a given constant.

Equation (2.3) is an incompressibility condition (conservation of volume). The normal force balance on the droplet boundary $\partial\Omega(t)$ is given in (2.4). Note that the Young-Laplace condition is perturbed in this model by an active force, $f(\nabla\phi \cdot n)n$ which is locally controlled by the normal derivative of the concentration of an internal solute, ϕ. The kinematic condition states that the normal velocity of the sharp interface, V, is given by the normal velocity of the fluid on $\partial\Omega(t)$. In the convection-diffusion dynamics given in (2.6), the total (convective + diffusive) solute flux is $j = (1-a)(-\nabla P)\phi - \nabla\phi$. In (2.7), we impose zero solute flux on the moving boundary, i.e., $j \cdot n - V\phi = 0$, where we insert the kinematic condition, Eq. (2.5). Simply put, the solute is convected at a slower velocity than that of the fluid. Hence, its concentration decreases (increases) towards an advancing (retracting) front. Note that a similar coupled free boundary model of polarization, migration and deformation of a living cell has recently been introduced in [13].

The problem (2.3) – (2.7) possesses a unique radially symmetric solution of prescribed area and total solute concentration with both $P = P^*$ and $\phi = \phi^*$ being constant.

If we consider a small perturbation of ϕ and P around ϕ^* and P^*, that is

$$\phi(t,x,y) = \phi^* + \varepsilon\tilde{\phi}(t,x,y) + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad P(t,x,y) = P^* + \varepsilon\tilde{P}(t,x,y) + \mathcal{O}(\varepsilon^2),$$

then we have

$$\nabla\phi(t,x,y) = \varepsilon\nabla\tilde{\phi}(t,x,y) + \mathcal{O}(\varepsilon^2),$$

and

$$\phi(t,x,y)\nabla P(t,x,y) = \varepsilon\phi^*\nabla\tilde{P}(t,x,y) + \mathcal{O}(\varepsilon^2),$$

hence, at order $\mathcal{O}(\varepsilon)$, the boundary condition (2.7) writes

$$\nabla\tilde{\phi}(t,x,y) \cdot n = a\phi^*\nabla\tilde{P}(t,x,y) \cdot n,$$

which yields the problem (1.1).

2.3 Biophysical meaning of the term $\beta f(V)$ for (1.1)

For each $t > 0$, we define the center of mass $\mathcal{C}_{\Omega(t)}$ of $\Omega(t)$ by

$$\mathcal{C}_{\Omega(t)} = \frac{1}{|\Omega(t)|}\int_{\Omega(t)} (x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{|\Omega_0|}\int_{\Omega(t)} (x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

by using the area preservation (1.3).

The velocity $u_{\mathcal{C}}(t)$ of the center of mass of $\Omega(t)$ is given by

$$u_{\mathcal{C}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{\Omega(t)}.$$

From incompressibility $(1.1)_1$ and boundary condition $(1.1)_2$, we deduce that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega(t)} x\,\mathrm{d}x\,\mathrm{d}y &= \int_{\partial\Omega(t)} xV\,\mathrm{d}\sigma = -\int_{\partial\Omega(t)} x\nabla P\cdot n\,\mathrm{d}\sigma \\ &= -\int_{\Omega(t)} \mathrm{div}\,(x\nabla P)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\Omega(t)} \nabla P\cdot\nabla x\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_{\Omega(t)} \mathrm{div}\,(P\nabla x)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\partial\Omega(t)} P\nabla x\cdot n\,\mathrm{d}\sigma \\ &= -\int_{\partial\Omega(t)} (\gamma\kappa+\beta f(V))\,n_x\,\mathrm{d}\sigma\end{aligned}$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega(t)} y\,\mathrm{d}x\,\mathrm{d}y = -\int_{\partial\Omega(t)} (\gamma\kappa+\beta f(V))\,n_y\,\mathrm{d}\sigma$$

Using the fact that $\int_{\partial\Omega(t)}\kappa n\,\mathrm{d}\sigma = 0$, it follows that

$$u_{\mathcal{C}}(t) = -\frac{\beta}{|\Omega_0|}\int_{\partial\Omega(t)} f(V)n\,\mathrm{d}\sigma\,. \tag{2.8}$$

We recognize that (2.8) represents the external force balance on the droplet $\Omega(t)$. This justifies the model which describes the deformation of the cell membrane under the action of an active force modeled by $f(V)n$ and describing the activity of the cytoskeleton, see [19, 12, 18] for a more precise biological description.

3 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4 and gives a constructive proof of the existence of traveling wave solutions to (1.1) for a certain range of parameters. We note that this constructive approach can be used to find these traveling wave solutions numerically, see Figure 1.

Given $\gamma,\lambda,\beta>0$, we look for a set Ω, solution of (1.7), in the particular form:

$$\Omega = \{(x,y)\in\mathbb{R}^2\,;\, x_L<x<x_R,\ -h(x)<y<h(x)\}, \tag{3.1}$$

for some function $h(x)$ defined and positive on an interval (x_L,x_R) and satisfying:

$$\begin{cases} h(x_L)=h(x_R)=0, \\ h'(x_L)=+\infty, \\ h'(x_R)=-\infty. \end{cases} \tag{3.2}$$

We can further fix the invariance by translation in the e_x direction by requiring that

$$x_L<0<x_R, \quad h'(0)=0. \tag{3.3}$$

We also note that it is enough to prove the result when $\gamma = 1$ by replacing the coefficients β and λ by β/γ and λ/γ and the function f by $x \mapsto f(\gamma x)$.

Our first task is to write equation (1.7) when Ω is given by (3.1) in term of the function $h(x)$. We first notice that the tangent vector $\mathbf{t}$, the normal vector $\mathbf{n}$ and the mean-curvature κ are defined by

$$\mathbf{t} = -\frac{(1, h'(x))}{\sqrt{1 + (h'(x))^2}}, \quad \mathbf{n} = \frac{(-h'(x), 1)}{\sqrt{1 + (h'(x))^2}}, \quad \kappa = -\frac{h''(x)}{(1 + (h'(x))^2)^{3/2}}.$$

These three quantities can be written easily using the function $Y(x)$ defined by

$$Y(x) = \mathbf{e}_x \cdot \mathbf{n} = \mathbf{n}_x = -\frac{h'(x)}{\sqrt{1 + (h'(x))^2}}. \tag{3.4}$$

In particular, we have

$$\kappa = Y'(x)$$

which is consistent with Frenet's formula $n_x' = -\kappa\sqrt{1 + (h'(x))^2}\mathbf{t}_x$.

Equation (1.7) (with $\gamma = 1$) and condition (3.3) then reduce to the following initial

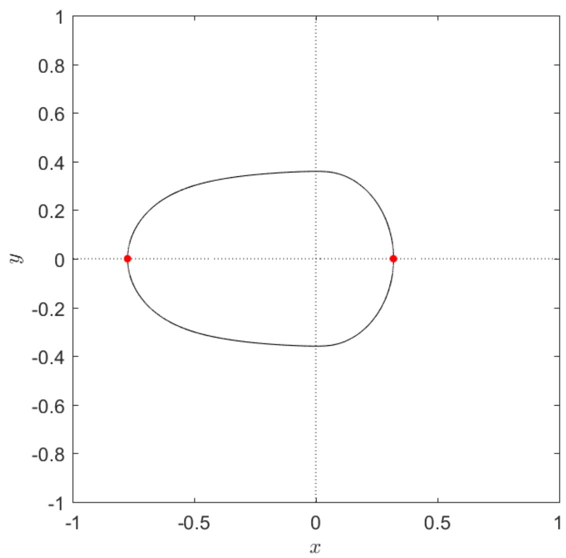

Figure 1: A shape of traveling wave solution Ω_0 defined by (3.1) for $\beta/\lambda = 4$ and $\gamma = 1$. The red dots indicate the point $x_L < 0$ on the left and $x_R > 0$ on the right. The function $h(x)$ is defined for $x \in [x_L, x_R]$ such that conditions (3.2) – (3.3) hold. The graph of h lies on the y-positve part of the plane, while the graph of $-h$ on the y-negative part.

value problem:

$$\begin{cases} Y'(x) = \lambda - cx + \beta f(c\, Y(x))), & \text{on } (x_L, x_R), \\ Y(0) = 0. \end{cases} \tag{3.5}$$

while the boundary conditions $(3.2)_{2,3}$ imply

$$Y(x_L) = -1 \quad \text{and} \quad Y(x_R) = +1. \tag{3.6}$$

Furthermore, for a given $Y(x)$, we can recover the function $h(x)$ by inverting the relation (3.4) to find

$$h'(x) = -\frac{Y(x)}{\sqrt{1 - Y^2(x)}}.$$

So if we can find $Y(x)$, solution of (3.5)-(3.6), we will define

$$h(x) = \int_{x_L}^{x} -\frac{Y(x)}{\sqrt{1 - Y^2(x)}}\, dx \qquad x \in [x_L, x_R]. \tag{3.7}$$

This function satisfies $h(x_L) = 0$, but the boundary condition $(3.2)_1$ requires the function Y to satisfy the additional condition:

$$\int_{x_L}^{x_R} -\frac{Y(x)}{\sqrt{1 - Y^2(x)}}\, \mathrm{d}x = 0. \tag{3.8}$$

We thus proceed as follows to construct the traveling wave solutions: We first show that when λ, β satisfy (3.9) and for some range of value of c, there exists Y satisfying (3.5)-(3.6). We will then show that there exists $c > 0$ such that (3.8) holds. This will lead to the following proposition:

Proposition 3.1. *Given $\lambda, \beta > 0$ such that*

$$\beta \lambda f'(0) > 1, \tag{3.9}$$

there exists $c > 0$, x_L and x_R such that the solution $Y(x)$ of (3.5) satisfies the conditions (3.6) and (3.8). Furthermore, for a fixed λ, the speed c converges to 0 when β approaches the critical value $\beta^* = \frac{1}{\lambda f'(0)}$.

The proof of Proposition 3.1 will occupy the rest of this section, but we can already show that it implies Theorem 1.4:

Proof of Theorem 1.4. With $Y(x)$ given by Proposition 3.1, we define $h(x)$ by (3.7). Condition (3.8) implies that $h(x_L) = h(x_R) = 0$ and we define the set Ω_β by (3.1). The boundary conditions (3.6) implies that the normal vector is continuous (it achieves the values $(-1, 0)$ and $(1, 0)$ continuously at the extremal points $(x_L, 0)$ and $(x_R, 0)$). This means that $\partial\Omega_\beta$ is C^1.

Furthermore, we have $\kappa(x) = Y'(x)$ and so Proposition 3.6 implies that

$$0 \le \kappa \le \lambda + \beta + c\beta f'(0) \text{ on } \partial\Omega_\beta$$

which implies that Ω_β is convex and that $\partial\Omega_\beta$ is $C^{1,1}$. In turns, (3.5) can be used to show that Y', and therefore κ is Lipschitz continuous so that the boundary $\partial\Omega_\beta$ is $C^{2,1}$ and satisfies (1.7).

Finally, it is readily seen that $h'(0) = 0$ (since $Y(0) = 0$) and (3.5) implies that the mean-curvature of $\partial\Omega_\beta$ at the point $(0, h(0))$ is given by $Y'(0) = \lambda$. When $\beta \to \beta^*$, we have $c \to 0$ and so Y converges to the solution of $Y' = \lambda$. In particular Ω_β converges to the set with constant mean curvature λ, that is the ball $B_{1/\lambda}$. □

The remainder of this section is devoted to the proof of Proposition 3.1. We first find the set of parameters for which the points x_L and x_R satisfying the condition (3.6) exist. In particular, we prove that the existence of x_L depends on λ and β, while for x_R the parameter c is also involved. This is done in Section 3.1. We then fix the parameters λ and β such that x_L exists and prove, in Section 3.2, that there exists a value $c^* > 0$ (depending on λ and β) such that x_R exists and condition (3.8) is verified.

Remark 3.2. *If we are looking for a traveling wave with $c > 0$, we must have $\lambda \ge 0$. Indeed, if $\lambda < 0$, then $Y'(0) < 0$ and the boundary conditions (3.6) imply that there exists two points $x_1 \in (x_L, 0)$ and $x_2 \in (0, x_R)$ such that $Y(x_1) > 0$, $Y(x_2) < 0$ and $Y'(x_1) = Y'(x_2) = 0$. Equation (3.5) then gives:*

$$cx_1 - \beta f(cY(x_1)) = cx_2 - \beta f(cY(x_2)) = \lambda,$$

which is a contradiction since the first term is negative and the second term is positive.

3.1 Proof of Proposition 3.1, part I: Existence of x_L and x_R

In this section, we prove two propositions concerning the existence of x_L and x_R.

The first proposition below gives the existence of x_L for all $c \ge 0$:

Proposition 3.3. *Let $\lambda, \beta > 0$ be such that*

$$\beta\lambda f'(0) \ge 1. \tag{3.10}$$

For all $c \ge 0$, there exists a point $x_L < 0$ such that the solution of (3.5) satisfies $-1 < Y(x) < 0$ for all $x \in (x_L, 0)$ and $Y(x_L) = -1$. Moreover, x_L is such that

$$-\beta f'(0) < x_L < 0, \tag{3.11}$$

and we have

$$Y'(x) \ge \frac{1}{\beta f'(0)} \quad \forall x \in [x_L, 0]. \tag{3.12}$$

The second proposition concerns the existence of x_R:

Proposition 3.4. *Let $\lambda, \beta > 0$ be such that (3.10) holds. There exists $c_{\max} \in [\frac{\lambda^2}{2}, \frac{(\lambda+\beta)^2}{2}]$ such that for all $c \in [0, c_{\max}]$, there exists a point $x_R > 0$ such that the solution of (3.5) satisfies $0 < Y(x) < 1$ for all $x \in (0, x_R)$ and $Y(x_R) = 1$. Furthermore, we have*

$$\begin{cases} Y'(x_R) > 0, & \text{if } c < c_{\max}, \\ Y'(x_R) = 0, & \text{if } c = c_{\max} \end{cases} \tag{3.13}$$

and

$$Y'(x) \geq \min\left\{\frac{1}{\beta f'(0)}, Y'(x_R)\right\} \quad \forall x \in [0, x_R]. \tag{3.14}$$

The proof of this proposition will also show that $x_R \leq \frac{\lambda+\beta}{c}$ for all $c > 0$ and $x_R \leq \frac{\lambda}{2}$ when $c \leq \frac{\lambda^2}{2}$, though such bounds will not be used later on.

We now turn to the proof of these two propositions.

Proof of Proposition 3.3. Let

$$x_L = \inf\{a < 0\,;\, Y(x) \in (-1, 0) \text{ for all } x \in (a, 0)\}. \tag{3.15}$$

Since $Y'(0) = \lambda > 0$, we see that $x_L < 0$ and possibly $x_L = -\infty$ (if $Y(x) > -1$ for all $x < 0$). We need to show that $x_L > -\infty$ and that $Y(x_L) = -1$. When $c = 0$, we have $Y(x) = \lambda x$ and the result is trivial. We thus assume that $c > 0$.

The convexity of $f(y)$ for $y < 0$ (Assumption (A4)) implies that

$$f(y) > f'(0)\, y \quad \text{for all } y < 0,$$

hence

$$f(cY(x)) > f'(0)\, cY(x) \quad \text{for all } x \in (x_L, 0).$$

From (3.5) it follows that

$$Y'(x) > \lambda - c\,x + \beta f'(0) cY(x) \quad \text{for all } x \in (x_L, 0). \tag{3.16}$$

Denoting $\mu = c\beta f'(0) > 0$, we rewrite (3.16) as $(e^{-\mu x}\, Y(x))' > (\lambda - c\,x)e^{-\mu x}$ and using the condition $Y(0) = 0$ we deduce that

$$Y(x) < \left(\frac{\lambda}{\mu} - \frac{c}{\mu^2}\right)(e^{\mu x} - 1) + \frac{c}{\mu}x, \quad \text{for all } x \in (x_L, 0). \tag{3.17}$$

Recalling the assumption (3.10) we see that $\frac{\lambda}{\mu} - \frac{c}{\mu^2} \geq 0$. Since $e^{\mu x} - 1 < 0$ for $x < 0$, it follows that

$$Y(x) < \frac{c}{\mu}x, \quad \text{for all } x \in (x_L, 0). \tag{3.18}$$

This implies that $Y(x_L) < 0$. And since we must have $Y(x_L) \geq -1$ it follows that $x_L > -\frac{\mu}{c} = -\beta f'(0)$ and $Y(x_L) = -1$.

To prove (3.12), we note that the function $V(x) = Y'(x)$ solves

$$\begin{cases} V'(x) = -c + c\beta f'(cY(x))V(x), \\ V(0) = \lambda \geq \frac{1}{\beta f'(0)}. \end{cases} \tag{3.19}$$

Since $f'(cY(x)) \leq f'(0)$, we see that, as long as $V \geq 0$, we have

$$\frac{d}{dx}\left[V(x) - \frac{1}{\beta f'(0)}\right] \leq c\beta f'(0)\left[V(x) - \frac{1}{\beta f'(0)}\right]$$

and a simple computation shows that $V(x) \geq \frac{1}{\beta f'(0)}$ for all $x \leq 0$. Inequality (3.12) follows. □

Next, we prove Proposition 3.4:

Proof of Proposition 3.4. We first prove the following lemma:

Lemma 3.5. *Let $\lambda > 0$ and $\beta > 0$ be given. For any $c > 0$ there exists $0 < \bar{x} < \frac{\lambda+\beta}{c}$ such that the solution $Y(x)$ of (3.5) is increasing on $(0, \bar{x})$ and decreasing on $(\bar{x}, \infty)$. Furthermore, we have*

$$Y(\bar{x}) = \sup_{x\in(0,\infty)} Y(x) \leq \frac{(\lambda+\beta)^2}{2c}. \tag{3.20}$$

Proof of Lemma 3.5. Equation (3.5) implies $Y'(0) = \lambda > 0$ and since $f(y) \leq 1$ for all $y \in \mathbb{R}$ it also gives

$$Y'(x) \leq \lambda - cx + \beta \qquad \text{for all } x \in \mathbb{R}$$

hence $Y'(x) < 0$ for $x > \frac{\lambda+\beta}{c}$. Since $Y(0) = 0$ this equation also gives $Y(x) \leq (\lambda+\beta)x - c\frac{x^2}{2}$ which has a maximum at $x = \frac{\lambda+\beta}{c}$. This implies the bound (3.20).

Recall that $V(x) = Y'(x)$ solves (3.19). Since $V(0) = \lambda > 0$, $V(x) = Y'(x) < 0$ for $x > \frac{\lambda+\beta}{c}$ and V is a continuous function, there exists a point $\tilde{x} > 0$ such that $V(\tilde{x}) = 0$. Moreover, whenever $V(\tilde{x}) = 0$ we have that $V'(\tilde{x}) = -c < 0$. This implies that V can only change sign once, the existence of $\bar{x}$ follows and $\bar{x} = \tilde{x}$. □

In view of Lemma 3.5, x_R exists such that $Y(x_R) = 1$ and $Y'(x_R) > 0$ if and only if $Y(\bar{x}) > 1$ and in such a case we always have

$$x_R < \bar{x}. \tag{3.21}$$

We thus start by showing that $Y(\bar{x}) > 1$ for $c \leq \frac{\lambda^2}{2}$. For this, we define

$$x_R = \sup\{a > 0\,;\, Y(x) \in (0,1) \text{ for all } x \in (0,a)\}, \tag{3.22}$$

with possibly $x_R = +\infty$. Since $Y(0) = 0$ and $Y'(0) = \lambda > 0$, we note that $x_R > 0$. Since $f(y) > 0$ for $y > 0$, (3.5) implies

$$Y'(x) > \lambda - cx \qquad \forall x \in (0, x_R)$$

and so $Y(x) > \lambda x - \frac{c}{2}x^2$. When $c \le \frac{\lambda^2}{2}$ this yields

$$Y(x) > \lambda x - \frac{\lambda^2}{4}x^2 = \lambda x(1 - \frac{\lambda}{4}x)$$

The right hand side is non negative for $x \in (0, \frac{4}{\lambda})$ and equal to 1 for $x = 2/\lambda$, so we must have $x_R < \lambda/2$ and $Y(x_R) = 1$ and therefore $Y(\bar{x}) > 1$.

We can now define

$$c_{\max} = \sup\{c_0\,;\, Y(\bar{x}) > 1 \text{ for all } c \in [0, c_0)\}, \tag{3.23}$$

which satisfies $c_{\max} \ge \frac{\lambda^2}{2}$. Note that (3.20) implies $c_{\max} \le \frac{(\lambda+\beta)^2}{2}$.

Finally, by continuity with respect to c, we notice that when $c = c_{\max}$, we have $Y(\bar{x}) \ge 1$. Moreover, if $Y(\bar{x}) > 1$, then there exists $\delta > 0$ such that $\sup Y > 1$ for $c \in [c_{\max}, c_{\max} + \delta)$ which contradicts the definition of $c_{\max}$. Consequently, $Y(\bar{x}) = 1$ when $c = c_{\max}$ and so $x_R = \bar{x}$ and $Y'(x_R) = 0$.

Inequality (3.14) follows from (3.19), which implies that $V = Y'$ is decreasing whenever $V \le \frac{1}{\beta f'(0)}$ and so if $V(x_0) < \min\left\{\frac{1}{\beta f'(0)}, Y'(x_R)\right\}$ for some $x_0 < x_R$, then $V(x_R) \le V(x_0) < Y'(x_R) = V(x_R)$, a contradiction.

□

Proposition 3.6. *Let λ, β and c be such that x_L and x_R (given by Propositions 3.3 and 3.4) exist. Then $Y(x)$ satisfies*

$$0 \le Y'(x) \le \lambda + \beta + c\beta f'(0) \quad \forall x \in (x_L, x_R).$$

Proof. Lemma 3.5 and (3.21) implies that $Y'(x) \ge 0$ on $[0, x_R]$. Together with (3.12), this implies that $Y'(x) \ge 0$ on (x_L, x_R). Next, Equation (3.5) implies

$$Y'(x) \le \lambda - cx_L + \beta$$

and the upper bound follows from (3.11). □

3.2 Proof of Proposition 3.1, part II: The velocity c

Throughout this section, we fix λ and β such that

$$\beta\lambda f'(0) \ge 1,$$

and we denote by $Y(x,c)$, $x_L(c)$, $x_R(c)$ the solution of (3.5), (3.6) for all $c \in (0, c_{\max})$. To prove Proposition 3.1, we must show that if $\beta\lambda f'(0) > 1$, there exists $c \in (0, c_{\max})$ such that (3.8) is satisfied. We thus introduce the function

$$G(c) := \int_{x_L(c)}^{x_R(c)} \frac{Y(x,c)}{\sqrt{1 - Y^2(x,c)}} \, \mathrm{d}x, \tag{3.24}$$

and prove:

Proposition 3.7. *The function $G : [0, c_{\max}) \to \mathbb{R}$ defined by (3.24) is continuous and satisfies*

$$G(0) = 0, \qquad G(c) \to +\infty, \qquad \textit{as} \qquad c \to c_{max}.$$

Furthermore, when $\beta\lambda f'(0) > 1$, then

$$G(c) < 0 \quad \textit{for} \quad 0 < c \ll 1,$$

while when $\beta\lambda f'(0) = 1$ we have

$$G(c) > 0 \quad \textit{for all } c > 0.$$

Proposition 3.1 now follows from the following corollary:

Corollary 3.8. *Given $\lambda > 0$ and for all $\beta > \frac{1}{\lambda f'(0)}$, there exists $c_\beta \in (0, c_{\max})$ such that $G(c_\beta) = 0$. Furthermore, $c_\beta \to 0$ as $\beta \to \beta^*$.*

Proof of Corollary 3.8. The corollary is an immediate consequence of Proposition 3.7 since for any $\beta > \beta^*$ the intermediate value theorem gives the existence of $c_\beta > 0$ such that $G(c_\beta) = 0$. In order to show that $c_\beta \to 0$ as $\beta \to \beta^*$, we note that if we consider G as a function of β and c (instead of only c), then the continuity with respect to β can be proved similarly to that with respect to c. It follows that when $\beta \to \beta^*$ along any subsequence such that $c_\beta \to c^*$ (such a subsequence exists since $0 \le c \le c_{max} \le \frac{(\lambda+\beta)^2}{2}$) we have $0 = G(\beta, c_\beta) \to G(\beta^*, c^*)$. Since $G(\beta^*, c) > 0$ for $c > 0$, it follows that $c^* = 0$ and that the whole sequence c_β converges to zero. □

Proof of Proposition 3.7. Continuity of $c \mapsto G(c)$. Differentiating equation (3.5) with respect to c, we find that the function $Z : x \mapsto \partial_c Y(x,c)$ solves

$$\begin{cases} Z'(x) = -x + \beta f'(cY(x))[Y(x) + cZ(x)], & \text{on } (x_L, x_R), \\ Z(0) = 0. \end{cases} \tag{3.25}$$

Moreover using that $0 \le f'(y) \le f'(0)$ for all $y \in \mathbb{R}$ and that $|Y| \le 1$, we deduce that there exists a constant C such that for all $x \in \mathbb{R}$ and for all $c \in (0, c_{max})$

$$|\partial_c Y(x,c)| \le C.$$

Since $x_L(c)$ and $x_R(c)$ are determined by the conditions

$$Y(x_L, c) = -1 \quad \text{and} \quad Y(x_R, c) = 1,$$

and recalling (3.12) and (3.13), we can thus apply the implicit function theorem to get that $c \mapsto x_L(c)$ and $c \mapsto x_R(c)$ are continuous Lipschitz functions.

To prove the continuity of G, we now consider a sequence c_n of positive number such that $c_n \to c > 0$. We fix $\delta > 0$. The continuity of x_L and x_R implies that for large enough n:

$$x_L(c_n) \leq x_L(c) + \delta \leq x_L(c_n) + 2\delta,$$

and

$$x_R(c_n) \geq x_R(c) - \delta \geq x_R(c_n) - 2\delta.$$

Furthermore, since $Y(x, c_n) \to Y(x, c)$ uniformly in $[x_L(c), x_R(c)]$, we have

$$|Y(x, c_n)| \leq 1 - \eta \qquad \text{in } (x_L(c) + \delta, x_R(c) - \delta), \tag{3.26}$$

for some $\eta > 0$ and n large enough.

We now write

$$\begin{aligned} G(c_n) &= \int_{x_L(c_n)}^{x_R(c_n)} \frac{Y(x, c_n)}{\sqrt{1 - Y^2(x, c_n)}}\, \mathrm{d}x \\ &= \int_{x_L(c_n)+\delta}^{x_R(c_n)-\delta} \frac{Y(x, c_n)}{\sqrt{1 - Y^2(x, c_n)}}\, \mathrm{d}x + \int_{x_L(c_n)}^{x_L(c_n)+\delta} \frac{Y(x, c_n)}{\sqrt{1 - Y^2(x, c_n)}}\, \mathrm{d}x \\ &\quad + \int_{x_R(c_n)-\delta}^{x_R(c_n)} \frac{Y(x, c_n)}{\sqrt{1 - Y^2(x, c_n)}}\, \mathrm{d}x. \end{aligned}$$

The bound (3.26) implies that the first integral converges to $\int_{x_L(c)+\delta}^{x_R(c)-\delta} \frac{Y(x,c)}{\sqrt{1-Y^2(x,c)}}\, \mathrm{d}x$ when $n \to \infty$. Next, we see that for $x \in [x_L(c_n), x_L(c_n) + \delta]$, we have $|Y(x)| \leq 1$, $1 - Y(x) \geq 1$ and $1 + Y(x) \geq C(x - x_L(c_n))$ (using (3.12)). It follows that the second term satisfies

$$\left| \int_{x_L(c_n)}^{x_L(c_n)+\delta} \frac{Y(x, c_n)}{\sqrt{1 - Y^2(x, c_n)}}\, \mathrm{d}x \right| \leq \int_{x_L(c_n)}^{x_L(c_n)+\delta} \frac{1}{\sqrt{C(x - x_L(c_n))}}\, \mathrm{d}x \leq C' \delta^{1/2},$$

where C, C' are positive constants.

Using a similar bound for the third term (using (3.14)) yields that

$$\lim_{n \to \infty} G(c_n) = \int_{x_L(c)+\delta}^{x_R(c)-\delta} \frac{Y(x, c)}{\sqrt{1 - Y^2(x, c)}}\, \mathrm{d}x + \mathcal{O}(\delta^{1/2}) = G(c) + \mathcal{O}(\delta^{1/2})$$

from which we deduce the continuity of G.

Behavior of G for $c < c_{max}$. It is easy to check that

$$Y(x, 0) = \lambda x, \quad x_L(0) = -\frac{1}{\lambda}, \quad x_R(0) = \frac{1}{\lambda},$$

hence

$$G(0) = \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} \frac{\lambda x}{\sqrt{1-\lambda^2 x^2}} \, \mathrm{d}x = 0. \tag{3.27}$$

Next, we define the function $H : (-1,1) \to \mathbb{R}$ by $H(y) = \frac{y}{\sqrt{1-y^2}}$. Using (3.5) we have:

$$\begin{aligned} G(c) = \int_{x_L(c)}^{x_R(c)} H(Y(x,c)) \, \mathrm{d}x &= \frac{1}{\lambda} \int_{x_L(c)}^{x_R(c)} \lambda H(Y(x,c)) \, \mathrm{d}x \\ &= \frac{1}{\lambda} \Big[\int_{x_L(c)}^{x_R(c)} \partial_x Y(x,c) H(Y(x,c)) \, \mathrm{d}x \\ &\quad + \int_{x_L(c)}^{x_R(c)} \left(cx - \beta f(cY(x,c))\right) H(Y(x,c)) \, \mathrm{d}x \Big] \\ &= \frac{1}{\lambda} \int_{x_L(c)}^{x_R(c)} \left(cx - \beta f(cY(x,c))\right) H(Y(x,c)) \, \mathrm{d}x. \end{aligned}$$

For all $c \in [0, c_{\max})$, we know that

$$\begin{cases} H(Y(x,c)) > 0 & \text{for all } 0 < x < x_R(c), \\ H(Y(x,c)) < 0 & \text{for all } x_L(c) < x < 0, \end{cases} \tag{3.28}$$

so we need to determine the sign of the function $W(x) = cx - \beta f(cY(x,c))$.

When

$$\beta \lambda f'(0) = 1,$$

We use the fact that the function $W(x) = cx - \beta f(cY(x,c))$ solves

$$W'(x) = c - \beta c f'(cY(x)) Y'(x) = c - \beta c f'(cY(x))(\lambda - W)$$

and so

$$W'(x) - \beta c f'(cY(x)) W = c - \beta c f'(cY(x)) \lambda \geq c - \beta c f'(0) \lambda = 0.$$

We deduce

$$\begin{cases} W(x) > 0 & \text{for } x > 0 \\ W(x) < 0 & \text{for } x < 0 \end{cases}$$

which together with (3.28) implies that $G(c) > 0$ for all $c > 0$.

We now assume that

$$\beta \lambda f'(0) > 1$$

and want to prove that $G(c) < 0$ for $0 < c \ll 1$.

The concavity of $f(y)$ for $y > 0$ (Assumption (A4)) implies that $c \mapsto \frac{f(c)}{c}$ is monotone decreasing for $c > 0$. Indeed, the function $h(y) = yf'(y) - f(y)$ satisfies $h(0) = 0$ and $h'(y) = yf''(y) \leq 0$ for $y \geq 0$ and thus $h(y) \leq 0$ for $y > 0$. It follows that the function $g(y) = \frac{f(y)}{y}$, which satisfies $g'(y) = \frac{h(y)}{y^2}$, is decreasing for $y > 0$. We thus have $f(y) \geq \frac{f(c)}{c} y$ for all $0 \leq y \leq c$ and so

$$f(cY(x)) \geq f(c)Y(x) \qquad \forall x \in (0, x_R),$$

which implies

$$Y'(x) \geq \lambda - c\,x + \beta f(c)\,Y(x) \qquad \text{for all } x \in (0, x_R). \tag{3.29}$$

Denoting $\mu = \beta f(c) > 0$, we rewrite (3.29) as $(e^{-\mu\,x}\,Y(x))' > (\lambda - c\,x)e^{-\mu x}$ and using the condition $Y(0) = 0$ we deduce

$$Y(x) \geq \left(\frac{\lambda}{\mu} - \frac{c}{\mu^2}\right)(e^{\mu x} - 1) + \frac{c}{\mu} x, \quad \text{for all } x \in (0, x_R]. \tag{3.30}$$

We can now write

$$\begin{aligned} W(x) &\leq cx - \beta f(c) Y(x,c) \\ &\leq cx + \mu\left[-\left(\lambda - \frac{c}{\mu}\right)\frac{e^{\mu x} - 1}{\mu} - \frac{c}{\mu} x\right] = \frac{c}{\mu}\left(1 - \frac{\lambda\mu}{c}\right)(e^{\mu x} - 1)\,. \end{aligned}$$

We note that $1 - \frac{\lambda\mu}{c} = 1 - \lambda\beta\frac{f(c)}{c} \to 1 - \lambda\beta f'(0) < 0$ when $c \to 0$ and so $1 - \frac{\lambda\mu}{c} < 0$ for $c \ll 1$. We deduce

$$W(x) < 0, \qquad \text{for all } 0 < x < x_R(c) \text{ when } c \ll 1. \tag{3.31}$$

Next, we consider $x_L(c) < x < 0$. The convexity of $f(y)$ for $y < 0$ implies that $y \mapsto \frac{f(y)}{y}$ is increasing for $y < 0$ and so

$$f(cY(x,c)) < -f(-c)Y(x,c), \qquad \forall x \in (x_L(c), 0)$$

and by the upper bound (3.17) it follows that

$$\begin{aligned} W(x) = cx - \beta f(cY(x,c)) &> cx + \beta f(-c)\left[\left(\lambda - \frac{c}{\mu}\right)\frac{e^{\mu x} - 1}{\mu} + \frac{c}{\mu} x\right] \\ &= \frac{c}{\mu}\left[\mu + \beta f(-c)\right] x + \beta f(-c)\left(\lambda - \frac{c}{\mu}\right)\frac{e^{\mu x} - 1}{\mu}, \end{aligned}$$

where $\mu = c\beta f'(0) > 0$. We deduce

$$W(x) > \frac{1}{f'(0)}\left[cf'(0) + f(-c)\right] x + \frac{-f(-c)}{f'(0)}\left[\lambda\beta f'(0) - 1\right]\frac{1 - e^{\mu x}}{\mu}.$$

Recalling that $x < 0$ and that $cf'(0) + f(-c) > 0$ by the convexity of $f(y)$ for $y < 0$, we see that the first term is negative but it is of order c^2 when $c \ll 1$. The second term is positive since $-f(-c) > 0$, $\lambda\beta f'(0) - 1 > 0$ and $1 - e^{\mu x} > 0$ and of order c. More precisely, we can write:

$$\begin{aligned} W(x) &> \mathcal{O}(c^2|x|) + (c + \mathcal{O}(c^2)) \left[\lambda\beta f'(0) - 1\right] (-x + \mathcal{O}(cx^2)) \\ &> -c\left[\lambda\beta f'(0) - 1\right] x + \mathcal{O}(c^2|x|) + \mathcal{O}(c^2x^2). \end{aligned}$$

Using the fact that $\lambda\beta f'(0) - 1 > 0$ and the bound (3.11), we deduce that for c small enough,

$$W(x) > 0, \qquad \text{for all } x_L(c) < x < 0. \tag{3.32}$$

We can now conclude and give the behavior of G for $c \ll 1$: Inequalities (3.28), (3.31) and (3.32) imply that

$$\begin{cases} W(x)H(Y(x,c)) < 0 & \text{for } 0 < x < x_R(c), \\ W(x)H(Y(x,c)) < 0 & \text{for } x_L(c) < x < 0, \end{cases} \tag{3.33}$$

as long as $0 < c \ll 1$ and therefore $G(c) = \frac{1}{\lambda}\int_{x_L(c)}^{x_R(c)} W(x)H(Y(x,c))\,\mathrm{d}x < 0$.

Behavior of G when $c \to c_{\max}$. Proposition 3.4, (3.13) gives:

$$\partial_x Y(c_{\max}, x_R(c_{\max})) = 0.$$

Recalling that $V(x) = \partial_x Y(x,c)$ solves (3.19) we deduce that

$$Y(c_{\max}, x) = 1 - c_{\max}(x - x_R(c_{\max}))^2 + \mathcal{O}\left(|x - x_R(c_{\max})|^3\right) \qquad \text{as } x \to x_R. \tag{3.34}$$

Thus, we get that

$$\begin{aligned} 1 - Y^2(c_{\max}, x) &= 1 - 1 + 2c_{\max}(x - x_R(c_{\max}))^2 + \mathcal{O}\left(|x - x_R(c_{\max})|^3\right) \\ &= 2c_{\max}(x - x_R(c_{\max}))^2 + \mathcal{O}\left(|x - x_R(c_{\max})|^3\right), \end{aligned}$$

leading to

$$\sqrt{1 - Y^2(c_{\max}, x)} = \sqrt{2c_{\max}}\,|x - x_R(c_{\max})| + \mathcal{O}\left(|x - x_R(c_{\max})|^{3/2}\right). \tag{3.35}$$

We notice that for all $\varepsilon > 0$ the function G can be written by

$$\begin{aligned} G(c) &= \int_{x_L(c)}^{x_R(c)} \frac{Y(x,c)}{\sqrt{1 - Y^2(x,c)}}\,\mathrm{d}x \\ &= \int_{x_L(c)}^{x_R(c)-\varepsilon} \frac{Y(x,c)}{\sqrt{1 - Y^2(x,c)}}\,\mathrm{d}x + \int_{x_R(c)-\varepsilon}^{x_R(c)} \frac{Y(x,c)}{\sqrt{1 - Y^2(x,c)}}\,\mathrm{d}x. \end{aligned}$$

The first term in the right hand side is always finite, while for the second term, we use (3.34) and (3.35) for $c \to c_{\max}$ to get:

$$\int_{x_R(c)-\varepsilon}^{x_R(c)} \frac{Y(x,c)}{\sqrt{1 - Y^2(x,c)}}\,\mathrm{d}x \to \int_{x_R(c_{\max})-\varepsilon}^{x_R(c_{\max})} \frac{1}{\sqrt{2}\,|x - x_R(c_{\max})|}\,\mathrm{d}x = +\infty.$$

We deduce that

$$G(c) \to +\infty \text{ as } c \to c_{\max} \tag{3.36}$$

which completes the proof of Proposition 3.7. $\square$

4 Proof of Theorem 1.5

In this section we prove the existence of a branch of traveling wave solutions bifurcating from the family of radially symmetric steady states B_{R_0}. We choose β as the bifurcation parameter, while the volume of the cell and the surface tension γ are fixed.

Since the disk B_{R_0} is a solution of the equation (1.6) with zero velocity $c = 0$, we seek other solutions in the form of a perturbation of the disk of radius R_0:

$$\Omega_0 = \{(r,\theta) \,:\, 0 \le r < R_0 + \rho(\theta) \text{ and } \theta \in [-\pi, +\pi]\}, \tag{4.1}$$

where the function $\rho : \mathbb{R} \to (-R_0, \infty)$ is 2π-periodic and such that

$$\int_{-\pi}^{\pi} (R_0 + \rho(\theta))^2 - R_0^2 \, \mathrm{d}\theta = 0$$

(this condition guarantee that $|\Omega_0| = |B_{R_0}|$). Furthermore, since we look for traveling wave propagating in the x-direction, we restrict ourselves (as in the previous section) to domain Ω_0 that are symmetric with respect to the y-axis. We thus introduce the functional space:

$$X = \{\rho \in \mathcal{C}^{2,\alpha}_{\text{per}}(-\pi,\pi); \rho(\theta) = \rho(-\theta)\,,\ \forall \theta \in (-\pi,\pi)\}. \tag{4.2}$$

Note that the boundary $\partial\Omega_0$ is parametrized by

$$\Big((R_0 + \rho(\theta))\cos\theta, (R_0 + \rho(\theta))\sin\theta\Big) \quad \text{for } \theta \in [-\pi, +\pi],$$

so the normal vector is given by

$$n(\theta) = \frac{1}{((R_0+\rho(\theta))^2 + \rho'(\theta)^2)^{1/2}} \begin{pmatrix} (R_0+\rho(\theta))\cos\theta + \rho'(\theta)\sin\theta \\ (R_0+\rho(\theta))\sin\theta - \rho'(\theta)\cos\theta \end{pmatrix},$$

and the mean-curvature by

$$\kappa(\theta) = \frac{(R_0+\rho(\theta))^2 + 2\rho'(\theta)^2 - (R_0+\rho(\theta))\rho''(\theta)}{((R_0+\rho(\theta))^2 + \rho'(\theta)^2)^{3/2}}.$$

The equation (1.7) can thus be rewritten as

$$\gamma\kappa(\theta) - \beta f\left(c\frac{(R_0+\rho(\theta))\cos\theta + \rho'(\theta)\sin\theta}{((R_0+\rho(\theta))^2 + \rho'(\theta)^2)^{1/2}}\right) + c(R_0+\rho(\theta))\cos\theta = \lambda, \tag{4.3}$$

for all $\theta \in [-\pi, \pi)$. Therefore, the existence of a boundary $\partial\Omega_0$ solving (1.7) follows from the existence of a function ρ solution of equation (4.3).

The existence of a branch of non trivial traveling waves now follows from the following bifurcation theorem:

Theorem 4.1. *Assume that f satisfies assumptions of Theorem 1.5. There exists an interval $I = (-\delta, \delta)$ and four C^1 functions $\rho : I \to X$, $\beta : I \to \mathbb{R}$, $c : I \to \mathbb{R}$ and $\lambda : I \to \mathbb{R}$ such that*

(i) For all $s \in I$, the function $\theta \to \rho(s, \theta)$ is a solution of (4.3) with $\beta = \beta(s)$, $c = c(s)$ and $\lambda = \lambda(s)$.
(ii) The function $s \mapsto c(s)$ is such that $c(0) = 0$ and $c'(0) = 1$.
(iii) The function $s \mapsto \beta(s)$ is such that $\beta(0) = \frac{R_0}{f'(0)}$, $\beta'(0) = 0$ and $\beta''(0) > 0$.
(iv) The function $s \mapsto \lambda(s)$ is such that $\lambda(0) = \frac{\gamma}{R_0}$, $\lambda'(0) = 0$.

Proof of Theorem 4.1. First, we note that the original problem (1.6) is invariant by translation. Thus, it is natural to eliminate these invariances by looking for solution of (4.3) satisfying the orthogonality conditions $\int_{-\pi}^{\pi} \rho(\theta) \cos\theta \, \mathrm{d}\theta = 0$ (the invariance by translation in the y direction was eliminated by the symmetry assumption).

We recall that X is defined by (4.2) and we define

$$Y = \{\rho \in \mathcal{C}^{0,\alpha}_{\mathrm{per}}(-\pi, \pi); \rho(\theta) = \rho(-\theta)\,,\ \forall \theta \in (-\pi, \pi)\},$$

and the function

$$\mathcal{F} : \mathbb{R} \times X \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R} \times \mathbb{R}$$

by

$$\begin{aligned} \mathcal{F}(\beta, \rho, c, \lambda) = \Big(\gamma\kappa(\theta) - \beta f\left(c\frac{(R_0 + \rho(\theta))\cos\theta + \rho'(\theta)\sin\theta}{((R_0 + \rho(\theta))^2 + \rho'(\theta)^2)^{1/2}}\right) \\ + c(R_0 + \rho(\theta))\cos\theta - \lambda, \int_{-\pi}^{\pi} \left((R_0 + \rho(\theta))^2 - R_0^2\right) \mathrm{d}\theta, \\ \int_{-\pi}^{\pi} \rho(\theta)\cos\theta \, \mathrm{d}\theta\Big). \end{aligned} \tag{4.4}$$

The proof of the theorem relies on a series of properties of $\mathcal{F}$ that allow us to apply the local bifurcation Theorem A.1.

Lemma 4.2. *Assume that f satisfies assumptions (A1) – (A4) and let $\beta_0 = \frac{R_0}{f'(0)}$. Then the functional $\mathcal{F}$ defined by (4.4) has the following properties*

1. *$\mathcal{F}(\beta, 0, 0, \frac{\gamma}{R_0}) = 0$ for all $\beta \in \mathbb{R}$.*
2. *$\operatorname{Ker} \partial_{(\rho,c,\lambda)} \mathcal{F}(\beta_0, 0, 0, \frac{\gamma}{R_0})$ is a one dimensional subspace of $\mathbb{R} \times X \times \mathbb{R} \times \mathbb{R}$ spanned by $(0, 1, 0)$;*
3. *$\operatorname{Range} \partial_{(\rho,c,\lambda)} \mathcal{F}(\beta_0, 0, 0, \frac{\gamma}{R_0})$ is a closed subspace of $Y \times \mathbb{R} \times \mathbb{R}$ of codimension 1;*
4. *$\partial_\beta \partial_{(\rho,c,\lambda)} \mathcal{F}(\beta_0, 0, 0, \frac{\gamma}{R_0})[(0, 1, 0)] \notin \operatorname{Range} \partial_{(\rho,c,\lambda)} \mathcal{F}(\beta_0, 0, 0, \frac{\gamma}{R_0})$.*

Proof of lemma 4.2. The first point is obvious. Next, we compute $\mathcal{L}_\beta := \partial_{(\rho,c,\lambda)} \mathcal{F}(\beta, 0, 0, \frac{\gamma}{R_0})$ which is the linear operator

$$\mathcal{L}_\beta : X \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R} \times \mathbb{R}$$

defined by

$$\mathcal{L}_\beta(\rho,c,\lambda)=\mathcal{F}_\rho(\beta,0,0,\gamma/R_0)[\,\rho\,]+\mathcal{F}_c(\beta,0,0,\gamma/R_0)\,c+\mathcal{F}_\lambda(\beta,0,0,\gamma/R_0)\,\lambda. \tag{4.5}$$

We recall that the linear operator $\mathcal{F}_\rho(\beta,\rho,c,\lambda)$ is defined by

$$\mathcal{F}_\rho(\beta,\rho,c,\lambda)[\eta]=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\beta,\rho+\varepsilon\eta,c,\lambda)_{|\varepsilon=0},\quad \text{for } \eta\in X,$$

and we compute:

$$\begin{aligned}&\mathcal{F}(\beta,\rho+\varepsilon\eta,c,\lambda)\\&=\Big(\gamma\,\kappa_{\rho+\varepsilon\eta}(\theta)-\beta f\left(c\frac{(R_0+\rho(\theta)+\varepsilon\eta(\theta))\cos\theta+(\rho'(\theta)+\varepsilon\eta'(\theta))\sin\theta}{[(R_0+\rho(\theta)+\varepsilon\eta(\theta))^2+(\rho'(\theta)+\varepsilon\eta'(\theta))^2]^{1/2}}\right)\\&+c\,[R_0+\rho(\theta)+\varepsilon\eta(\theta)]\cos(\theta)-\lambda,\int_{-\pi}^{\pi}\left((R_0+\rho(\theta)+\varepsilon\eta(\theta))^2-R_0^2\right)\,\mathrm{d}\theta,\\&\int_{-\pi}^{\pi}[\rho(\theta)+\varepsilon\eta(\theta)]\cos\theta\;\mathrm{d}\theta\Big),\end{aligned}\tag{4.6}$$

where $\kappa_{\rho+\varepsilon\eta}(\theta)$ is the mean-curvature of the perturbed boundary, that is

$$\begin{aligned}&\kappa_{\rho+\varepsilon\eta}(\theta)\\=\;&\frac{[R_0+\rho(\theta)+\varepsilon\eta(\theta)]^2+2[\rho'(\theta)+\varepsilon\eta'(\theta)]^2-[R_0+\rho(\theta)+\varepsilon\eta(\theta)]\,[\rho''(\theta)+\varepsilon\eta''(\theta)]}{[\,(R_0+\rho(\theta)+\varepsilon\eta(\theta))^2+(\rho'(\theta)+\varepsilon\eta'(\theta))^2\,]^{3/2}}.\end{aligned}$$

We derive the expression (4.6) with respect to ε, we compute it for $\varepsilon=0$, and we then consider $\rho=0$, $c=0$ and $\lambda=\frac{\gamma}{R_0}$. For $\eta=\rho$, we find the following expression

$$\begin{aligned}&\mathcal{F}_\rho(\beta,0,0,\gamma/R_0)[\,\rho\,]\\&=\left(-\gamma\,\frac{\rho(\theta)+\rho''(\theta)}{R_0^2},\int_{-\pi}^{\pi}2R_0\rho(\theta)\;\mathrm{d}\theta,\int_{-\pi}^{\pi}\rho(\theta)\cos\theta\;\mathrm{d}\theta\right).\end{aligned}\tag{4.7}$$

The second and the third terms in (4.5) are simpler to compute since c and λ are scalar quantities. We obtain that

$$\mathcal{F}_c(\beta,0,0,\gamma/R_0)\,c=\left(-c\,\beta f'(0)\cos\theta+c\,R_0\cos\theta,0,0\right), \tag{4.8}$$

and

$$\mathcal{F}_\lambda(\beta,0,0,\gamma/R_0)\,\lambda=(-\lambda,0,0)\,. \tag{4.9}$$

Finally, the linear operator $\mathcal{L}_\beta$ is given by the sum of the expressions (4.7) – (4.9), that is

$$\begin{aligned}\mathcal{L}_\beta(\rho,c,\lambda)=\Big(&-\gamma\,\frac{\rho(\theta)+\rho''(\theta)}{R_0^2}-c\beta f'(0)\cos\theta+cR_0\cos\theta-\lambda,\\&2R_0\int_{-\pi}^{\pi}\rho(\theta)\,d\theta,\int_{-\pi}^{\pi}\rho(\theta)\cos\theta\;\mathrm{d}\theta\Big).\end{aligned}\tag{4.10}$$

When $\beta_0 = \frac{R_0}{f'(0)}$, we get

$$\mathcal{L}_{\beta_0}(\rho, c, \lambda) = \left(-\gamma \frac{\rho(\theta) + \rho''(\theta)}{R_0^2} - \lambda, 2R_0 \int_{-\pi}^{\pi} \rho(\theta) \, \mathrm{d}\theta, \int_{-\pi}^{\pi} \rho(\theta) \cos\theta \, \mathrm{d}\theta \right).$$

Thus, the elements $\{(\rho, c, \lambda)\}$ belonging to $\operatorname{Ker} \mathcal{L}_{\beta_0}$ are such that

$$\rho''(\theta) + \rho(\theta) = \frac{-R_0^2}{\gamma}\lambda, \quad \int_{-\pi}^{\pi} \rho(\theta) \, \mathrm{d}\theta = 0, \quad \int_{-\pi}^{\pi} \rho(\theta) \cos\theta \, \mathrm{d}\theta = 0. \tag{4.11}$$

The parameter c does not appear in (4.11), so $(0, c, 0) \in \operatorname{Ker} \mathcal{L}_{\beta_0}$ for all $c \in \mathbb{R}$ and thus $\dim \operatorname{Ker} \mathcal{L}_{\beta_0} \geq 1$. Furthermore, if (ρ, λ) solves (4.11), then

$$\rho(\theta) = a\cos\theta + b\sin\theta - \frac{R_0^2}{\gamma}\lambda, \qquad \text{for some } a, b \in \mathbb{R}$$

and the conditions $\int_{-\pi}^{+\pi} \rho(\theta) \, \mathrm{d}\theta = 0$, $\int_{-\pi}^{+\pi} \rho(\theta) \cos\theta \, \mathrm{d}\theta = 0$ and $\rho(\theta) = \rho(-\theta)$ imply respectively $\lambda = 0$, $a = 0$ and $b = 0$, hence $\operatorname{Ker} \mathcal{L}_{\beta_0} = \operatorname{span}\{(0, 1, 0)\}$ and $\dim \operatorname{Ker} \mathcal{L}_{\beta_0} = 1$.

Next, we show that the range of $\mathcal{L}_{\beta_0}$ consists of all the triplets $(f, C_1, C_2) \in Y \times \mathbb{R} \times \mathbb{R}$ such that $\int_{-\pi}^{\pi} f(\theta) \cos\theta \, \mathrm{d}\theta = 0$. The fact that this condition is necessary is obtained by multiplying the equation

$$-\gamma \frac{\rho(\theta) + \rho''(\theta)}{R_0^2} - \lambda = f \tag{4.12}$$

by $\cos(\theta)$ and integrating over $(-\pi, \pi)$. To check that this condition is sufficient, we note that given $f \in \mathcal{C}^{0,\alpha}_{\text{per}}(-\pi, \pi)$ and $\lambda \in \mathbb{R}$, equation (4.12) has a solution in $\mathcal{C}^{2,\alpha}_{\text{per}}(-\pi, \pi)$ if and only if $\int_{-\pi}^{\pi} f(\theta) \cos\theta \, \mathrm{d}\theta = \int_{-\pi}^{\pi} f(\theta) \sin\theta \, \mathrm{d}\theta = 0$, where the second condition is always satisfied for $f \in Y$ (f is even). The general solution of (4.12) is then of the form

$$\rho(\theta) = \bar{\rho}(\theta) + a\cos\theta + b\sin\theta$$

for some particular solution $\bar{\rho}$, which we can assume to be even (otherwise we replace it with $\frac{1}{2}[\bar{\rho}(\theta) + \bar{\rho}(-\theta)]$). We now take $b = 0$ (so $\rho \in X$) and choose a such that $\int_{-\pi}^{\pi} \rho(\theta) \cos\theta \, \mathrm{d}\theta = C_2$. Finally, integrating (4.12) with respect to θ yields

$$-\frac{\gamma}{R_0^2} \int_{-\pi}^{\pi} \rho(\theta) \, \mathrm{d}\theta - 2\pi\lambda = \int_{-\pi}^{\pi} f(\theta) \, \mathrm{d}\theta$$

and so for an appropriate choice of λ we will have $2R_0 \int_{-\pi}^{\pi} \rho(\theta) \, \mathrm{d}\theta = C_2$.

Lastly, we have to prove the transversality condition (4) with respect to the value β_0, that is we have to prove that $(\partial_\beta \mathcal{L}_{\beta_0})(0, 1, 0) \notin \operatorname{Range} \mathcal{L}_{\beta_0}$. We have

$$(\partial_\beta \mathcal{L}_{\beta_0})(0, 1, 0) = (-f'(0)\cos\theta, 0, 0).$$

Assume by contradiction that $(\partial_\beta \mathcal{L}_{\beta_0})(0,1,0) \in \mathrm{Range}\, \mathcal{L}_{\beta_0}$, then we would have that

$$-\gamma \frac{\rho(\theta)+\rho''(\theta)}{R_0^2} - \lambda = f'(0)\cos\theta. \tag{4.13}$$

Multiplying by $\cos\theta$ and integrating on $[-\pi,+\pi]$, we get (integrating by parts and using the fact that $\int_{-\pi}^{\pi}\rho(\theta)\cos\theta\, \mathrm{d}\theta = 0$)

$$0 = f'(0)\int_{-\pi}^{\pi}\cos^2\theta\, \mathrm{d}\theta$$

which is a contradiction since $f'(0)>0$. $\square$

We can now apply the bifurcation theorem A.1: Let us denote by Z any complement space of $\mathrm{Ker}\, \mathcal{L}_{\beta_0}$, there exists an interval $I=(-\delta,\delta)$ and four $\mathcal{C}^1$ functions $\beta : I \to \mathbb{R}$, $\rho : I\times[-\pi,+\pi]\to Z$, $c : I \to Z$ and $\lambda : I \to Z$ such that

$$\mathcal{F}(\beta(s),\rho(s,\theta),c(s),\lambda(s)) = (0,0,0) \quad \text{for all } s\in I,\ \theta\in[-\pi,+\pi], \tag{4.14}$$

and

$$\begin{cases} \beta(0) &= \beta_0 = \frac{R_0}{f'(0)}, \\ \rho(0,\theta) &= 0 \quad \text{and} \quad \partial_s\rho(0,\theta) = 0 \qquad \text{for all } \theta\in[-\pi,+\pi], \\ c(0) &= 0 \quad \text{and} \quad c'(0) = 1, \\ \lambda(0) &= \frac{\gamma}{R_0} \quad \text{and} \quad \lambda'(0) = 0. \end{cases} \tag{4.15}$$

This completes the proof of points (i) and (ii) of Theorem 4.1. The rest of this section is devoted to proving Lemma 4.3 and 4.4 below which together imply Theorem 4.1-(iii) (some of the more technical computations are presented in the appendix).

In the proofs that follow, we use extensively the fact that the functions $\theta\mapsto\rho(0,\theta)$ and $\theta\mapsto\partial_s\rho(0,\theta)$ (and all their derivatives with respect to θ) vanish. Using the fact that $\int_{-\pi}^{\pi}\rho(s,\theta)\cos\theta\, \mathrm{d}\theta = 0$ and $\int_{-\pi}^{\pi}\left((R_0+\rho(s,\theta))^2 - R_0^2\right)\mathrm{d}\theta = 0$ for all s, we also have that $\partial_{ss}\rho(0,\theta)$ satisfies:

$$\int_{-\pi}^{\pi}\partial_{ss}\rho(0,\theta)\cos\theta\, \mathrm{d}\theta = 0, \quad \int_{-\pi}^{\pi}\partial_{ss}\rho(0,\theta)\sin\theta\, \mathrm{d}\theta, \quad \int_{-\pi}^{\pi}\partial_{ss}\rho(0,\theta)\, \mathrm{d}\theta = 0. \tag{4.16}$$

We start with:

Lemma 4.3. *We have $\beta'(0)=0$, $\lambda''(0)=0$ and $\partial_{ss}\rho(0,\theta)=0$ for all $\theta\in[-\pi,\pi]$.*

Proof of lemma 4.3. We differentiate with respect to s the first component of $\mathcal{F}$ and since (4.14) holds for all $s\in I$ we get that

$$\begin{aligned} 0 =& \gamma\partial_s k(s,\theta) - \beta'(s)f(z(s,\theta)) - \beta(s)f'(z(s,\theta))\partial_s z(s,\theta) \\ &+ c'(s)(R_0+\rho(s,\theta))\cos\theta + c(s)\partial_s\rho(s,\theta)\cos\theta - \lambda'(s), \end{aligned} \tag{4.17}$$

where the functions k and z are defined by

$$k(s,\theta)=\frac{((R_0+\rho)^2+2\partial_\theta\rho^2-(R_0+\rho)\partial_{\theta\theta}\rho)(s,\theta)}{\big(((R_0+\rho)^2+\partial_\theta\rho^2)^{3/2}\big)(s,\theta)}. \tag{4.18}$$

and

$$z(s,\theta)=c(s)\frac{(R_0+\rho(s,\theta))\cos\theta+\partial_\theta\rho(s,\theta)\sin\theta}{(\sqrt{(R_0+\rho(s,\theta))^2+(\partial_\theta\rho(s,\theta))^2})}. \tag{4.19}$$

Using (4.15) we first see that

$$z(0,\theta)=0 \quad \text{for all } \theta\in[-\pi,+\pi],$$

Since $f(0)=0$ the coefficient of $\beta'(0)$ in (4.17) for $s=0$ vanishes and we do not get any information.

We differentiate equation (4.17) with respect to s and we get

$$\begin{aligned}0=&\gamma\partial_{ss}k(s,\theta)-\beta''(s)f(z(s,\theta))-2\beta'(s)f'(z(s,\theta))\partial_s z(s,\theta)\\&-\beta(s)f'(z(s,\theta))\partial_{ss}z(s,\theta)-\beta(s)f''(z(s,\theta))(\partial_s z(s,\theta))^2\\&+c''(s)(R_0+\rho(s,\theta))\cos\theta+2c'(s)\partial_s\rho(s,\theta)\cos\theta\\&+c(s)\partial_{ss}\rho(s,\theta)\cos\theta-\lambda''(s).\end{aligned} \tag{4.20}$$

Computing (4.20) for $s=0$ (using (4.15) and the fact that $f(0)=f''(0)=0$) we obtain

$$0=\gamma\partial_{ss}k(0,\theta)-2\beta'(0)f'(0)\partial_s z(0,\theta)-\beta(0)f'(0)\partial_{ss}z(0,\theta)+R_0c''(0)\cos\theta-\lambda''(0).$$

Using Lemma C.1 and the fact that $\beta(0)=R_0/f'(0)$, we deduce

$$\gamma\partial_{ss}k(0,\theta)=2\beta'(0)f'(0)\cos(\theta)+\lambda''(0). \tag{4.21}$$

Lemma B.1 now gives

$$\frac{\gamma}{R_0^2}L[\partial_{ss}\rho(0,\cdot)]=2\beta'(0)f'(0)\cos(\theta)+\lambda''(0). \tag{4.22}$$

where L denotes the operator

$$L[u]=-u-\partial_{\theta\theta}u.$$

Multiplying (4.22) by $\cos\theta$, integrating on $[-\pi,\pi]$, and using the fact that $L^*[\cos]=0$, we obtain $\int_{-\pi}^{\pi}2\beta'(0)f'(0)\cos^2\theta\ \mathrm{d}\theta=0$ which implies that $\beta'(0)=0$ since $f'(0)\neq 0$. Integrating (4.22) on $[-\pi,\pi]$, using the fact that $L^*[1]=-1$ and (4.16), we obtain $\int_{-\pi}^{\pi}\lambda''(0)\ \mathrm{d}\theta=0$, that is $\lambda''(0)=0$. In turn, (4.22) now implies that $L[\partial_{ss}\rho(0,\cdot)]=0$, that is $\partial_{ss}\rho(0,\theta)=C_1\cos(\theta)+C_2\sin(\theta)$. Using (4.16) we find $C_1=C_2=0$ and so $\partial_{ss}\rho(0,\theta)=0$. □

Finally, we prove:

Lemma 4.4. *We have $\beta''(0) > 0$.*

Proof of lemma 4.4. We differentiate (4.20) with respect to s

$$\begin{aligned} 0 =& \gamma\partial_{sss}k(s,\theta) - \beta'''(s)f(z(s,\theta)) - 3\beta''(s)f'(z(s,\theta))\partial_s z(s,\theta) \\ & - \beta'(s)[3f''(z(s,\theta))(\partial_s z(s,\theta))^2 + 3f'(z(s,\theta))\partial_{ss}z(s,\theta)] \\ & - \beta(s)[f'''(z(s,\theta))(\partial_s z(s,\theta))^3 + 3f''(z(s,\theta))\partial_s z(s,\theta)\,\partial_{ss}z(s,\theta) \\ & + f'(z(s,\theta))\partial_{sss}z(s,\theta)] + c'''(s)(R_0 + \rho(s,\theta))\cos\theta \\ & + 3c''(s)\partial_s\rho(s,\theta)\cos\theta + 3c'(s)\partial_{ss}\rho(s,\theta)\cos\theta \\ & + c(s)\partial_{sss}\rho(s,\theta)\cos\theta - \lambda'''(s). \end{aligned} \tag{4.23}$$

When $s = 0$, using the fact that $c(0) = 0$, $c'(0) = 1$, $\beta(0) = R_0/f'(0)$, $\beta'(0) = 0$, $z(0,\theta) = 0$, $\partial_s z(0,\theta) = \cos\theta$, $\partial_{ss}z(0,\theta) = c''(0)\cos\theta$ (see Lemma C.1), $\rho(0,\theta) = \partial_s\rho(0,\theta) = \partial_{ss}\rho(0,\theta) = 0$ and the assumptions $f(0) = 0$, $f''(0) = 0$, we find:

$$\begin{aligned} 0 =& \gamma\partial_{sss}k(0,\theta) - 3\beta''(0)f'(0)\cos\theta - \frac{R_0}{f'(0)}[f'''(0)\cos^3\theta + f'(0)\partial_{sss}z(0,\theta)] \\ & + c'''(0)R_0\cos\theta - \lambda'''(0). \end{aligned}$$

Finally, using the expression for $\partial_{sss}z(0,\theta)$ given by Lemma C.2 and the formula for $\partial_{sss}k(0,\theta)$ given by Lemma B.2, we get:

$$\frac{\gamma}{R_0^2}L[\partial_{sss}\rho(0,\cdot)] = 3\beta''(0)f'(0)\cos\theta + \frac{R_0 f'''(0)}{f'(0)}\cos^3\theta + \lambda'''(0). \tag{4.24}$$

Since $L^*[\cos] = 0$, we deduce:

$$\int_{-\pi}^{+\pi}\left(3\beta''(0)f'(0)\cos\theta + R_0\frac{f'''(0)}{f'(0)}\cos^3\theta + \lambda'''(0)\right)\cos\theta\;\mathrm{d}\theta = 0,$$

leading to

$$\beta''(0) = -\frac{R_0}{4}\frac{f'''(0)}{(f'(0))^2}.$$

The result follows since $f'''(0) < 0$. □

This completes the proof of Theorem 4.1.

□

5 Conclusion and biological consequences

Spontaneous symmetry breaking is a characteristic of living cells, the model (1.1) introduced in [6] and studied in this work accounts for this biological phenomenon. Indeed for large enough value of the parameter β ($\beta \geq R_0/f'(0)$), two different behaviors take place: a symmetric cell with a zero velocity or an asymmetric cell with a non zero velocity. From a biological viewpoint this means that the rest state (1.4) is destabilized through a bifurcation at $\beta = R_0/f'(0)$. In other words, the polarization-translation mode, which breaks the front-symmetry and leads to motility, is unstable for $\beta = R_0/f'(0)$.

As a conclusion, the model presented here is intended to be a highly simplified representation of the biological cell. The analyse performed in this work allows to prove that the models (2.1) and (2.3) – (2.7) are close to an unstable system of equations. The model (1.1) while mathematically unpleasant, describes an important feature of cell motility. Its main interest lies in its relative simplicity as it is expressed as a single free-boundary model. Since it accurately describes the instability allowing cells to move, a more in-depth mathematical analysis would be interesting and challenging due to the non-conventional boundary condition. We leave the question of the existence of a Lyapunov function for (1.1) as an open question.

A A theorem of Crandall-Rabinovitz

We recall here the classical bifurcation theorem of Crandall-Rabinovitz [5] that we used to prove our result. Given U, V two real Banach spaces and a continuous map $\mathcal{F} : \mathbb{R} \times U \to V$, the goal is to analyze the structure of the solution set

$$\mathcal{F}[\lambda, u] = 0, \quad (\lambda, u) \in \mathbb{R} \times U.$$

Theorem A.1 (Local bifurcation [5])**.** *Let U, V be Banach spaces, W a neighborhood of $(\lambda_0, 0)$ in $\mathbb{R} \times U$ and $\mathcal{F} : W \longrightarrow V$. Suppose that the following properties are satisfied*

1. $\mathcal{F}(\lambda, 0) = 0$ *for all λ in a neighborhood of λ_0;*
2. *The Fréchet partial derivatives $\partial_u \mathcal{F}, \partial_\lambda \mathcal{F}, \partial_{\lambda u} \mathcal{F}$ exist and are continuous;*
3. $\operatorname{Ker} \partial_u \mathcal{F}(\lambda_0, 0)$ *is a one dimensional subspace of U spanned by a nonzero vector $u_0 \in U$;*
4. $\operatorname{Range} \partial_u \mathcal{F}(\lambda_0, 0)$ *is a closed subspace of V of codimension 1;*
5. $\partial_{\lambda u} \mathcal{F}(\lambda_0, 0)[u_0] \notin \operatorname{Range} \partial_u \mathcal{F}(\lambda_0, 0)$.

If Z is any complement of $\operatorname{Ker} \partial_u \mathcal{F}(\lambda_0, 0)$ in U, then, there is a neighborhood N of $(\lambda_0, 0)$ in $\mathbb{R} \times U$, an interval $I = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and two continuous functions

$$\varphi : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}, \quad \psi : (-\varepsilon, \varepsilon) \longrightarrow Z$$

such that $\varphi(0) = \lambda_0$, $\psi(0) = 0$ and

$$\mathcal{F}^{-1}[0] \cap U = \{(\varphi(s), su_0 + s\psi(s)) : |s| < \varepsilon\} \cup \{(\lambda, 0) : (\lambda, 0) \in N\}.$$

If $\partial_{uu} \mathcal{F}$ is continuous then the functions φ and ψ are once continuously differentiable.

Remark A.2. *In theorem A.1, $(\lambda_0, 0)$ is a bifurcation point of the equation $\mathcal{F}(\lambda, u) = 0$ in the following sense: in a neighborhood of $(\lambda_0, 0)$, the set of solutions of $\mathcal{F}(\lambda, u) =$*

0 *consists of two curves* Γ_1 *and* Γ_2 *which intersect only at the point* $(\lambda_0, 0)$*;* Γ_1 *is the curve* $(\lambda_0, 0)$ *and* Γ_2 *can be parameterized as follows:*

$$\Gamma_2 : (\lambda(s), u(s)),\ |s| \text{ small };\ (\lambda(0), u(0)) = (\lambda_0, 0);\ u'(0) = u_0,\ \lambda'(0) \neq 0.$$

B Computation of $\partial_s k$, $\partial_{ss} k$ and $\partial_{sss} k$

Recall that $k(s, \theta)$ is defined by (4.18) for $s \in I$ and $\theta \in [-\pi, +\pi]$.

Lemma B.1. *Assume* $\rho(0, \theta) = \partial_s \rho(0, \theta) = 0$ *for all* $\theta \in [-\pi, +\pi]$*. Then*

$$\partial_s k(0, \theta) = 0$$

and

$$R_0^2 \partial_{ss} k(0, \theta) = -\partial_{ss} \rho(0, \theta) - \partial_{ss\theta\theta} \rho(0, \theta). \tag{B.1}$$

Proof. Define the functions

$$N(s, \theta) = ((R_0 + \rho)^2 + 2\partial_\theta \rho^2 - (R_0 + \rho)\partial_{\theta\theta}\rho)(s, \theta)$$
$$D(s, \theta) = \big(((R_0 + \rho)^2 + \partial_\theta \rho^2)^{3/2}\big)(s, \theta)$$

so that

$$k(s, \theta) = \frac{N(s, \theta)}{D(s, \theta)} \quad \text{and} \quad \partial_s k(s, \theta) = \left(\frac{(\partial_s N)D - N(\partial_s D)}{D^2}\right)(s, \theta) \tag{B.2}$$

with

$$(\partial_s N)(s, \theta) = (2(R_0 + \rho)\partial_s \rho + 4\partial_\theta \rho\, \partial_{s\theta} \rho - R_0 \partial_{s\theta\theta} \rho - \partial_s \rho\, \partial_{\theta\theta} \rho - \rho\, \partial_{s\theta\theta} \rho)(s, \theta),$$
$$(\partial_s D)(s, \theta) = (3(\partial_\theta \rho\, \partial_{s\theta} \rho + (R_0 + \rho)\partial_s \rho)\sqrt{(R_0 + \rho)^2 + (\partial_\theta \rho)^2}\,)(s, \theta).$$

Since the $\rho(0, \theta) = \partial_s \rho(0, \theta) = 0$ for all θ, we see that $(\partial_s N)(0, \theta) = (\partial_s D)(0, \theta) = 0$ and so $\partial_s k(0, \theta) = 0$.

Next, we define

$$n(s, \theta) = ((\partial_s N)D - N(\partial_s D))(s, \theta) \quad \text{and} \quad d(s, \theta) = D^2(s, \theta),$$

so that $\partial_s k(s, \theta) = \frac{n}{d}(s, \theta)$ and

$$\partial_{ss} k(s, \theta) = \left(\frac{(\partial_s n)d - n(\partial_s d)}{d^2}\right)(s, \theta), \tag{B.3}$$

where

$$(\partial_s n)(s, \theta) = ((\partial_{ss} N)D - N(\partial_{ss} D))(s, \theta) \quad \text{and} \quad (\partial_s d)(s, \theta) = (2D\partial_s D)(s, \theta),$$

with

$$(\partial_{ss}N)(s,\theta) = (2\partial_s\rho^2 + (2R_0 + 2\rho - \partial_{\theta\theta}\rho)\partial_{ss}\rho + 4(\partial_{s\theta}\rho)^2 \\ + 4\partial_\theta\rho\,\partial_{ss\theta}\rho - R_0\partial_{ss\theta\theta}\rho - 2\partial_s\rho\partial_{s\theta\theta}\rho - \rho\partial_{ss\theta\theta}\rho)(s,\theta),$$

and

$$(\partial_{ss}D)(s,\theta) = \big(3((R_0+\rho)^2 + (\partial_\theta\rho)^2)^{-1/2}((R_0+\rho)\partial_s\rho + \partial_\theta\rho\partial_{s\theta}\rho)^2 \\ + 3\sqrt{(R_0+\rho)^2 + (\partial_\theta\rho)^2}((\partial_s\rho)^2 + (R_0+\rho)\partial_{ss}\rho + (\partial_{s\theta}\rho)^2 + \partial_\theta\rho\partial_{ss\theta}\rho)\big)(s,\theta).$$

Using the fact that $\rho(0,\theta) = \partial_s\rho(0,\theta) = 0$ for all θ, we see that

$$\partial_{ss}N(0,\theta) = 2R_0\partial_{ss}\rho(0,\theta) - R_0\partial_{ss\theta\theta}\rho(0,\theta),$$

and

$$\partial_{ss}D(0,\theta) = 3R_0^2\partial_{ss}\rho(0,\theta),$$

hence (note that $N(0,\theta) = R_0^2$ and $D(0,\theta) = R_0^3$)

$$\partial_s n(0,\theta) \quad = \quad -R_0^4\partial_{ss}\rho(0,\theta) - R_0^4\partial_{ss\theta\theta}\rho(0,\theta),$$

and (recall that $(\partial_s D)(0,\theta) = 0$) $\partial_s d(0,\theta) = 0$. We deduce that $\partial_{ss}k(0,\theta) = \frac{\partial_s n}{D^2}(0,\theta) = \frac{-R_0^4\partial_{ss}\rho(0,\theta) - R_0^4\partial_{ss\theta\theta}\rho(0,\theta)}{R_0^6}$ and (B.1) follows.

□

Lemma B.2. *Assume $\rho(0,\theta) = \partial_s\rho(0,\theta) = \partial_{ss}\rho(0,\theta) = 0$ for all $\theta \in [-\pi,+\pi]$. Then*

$$R_0^2\partial_{sss}k(0,\theta) = -\partial_{sss}\rho(0,\theta) - \partial_{sss\theta\theta}\rho(0,\theta).$$

Proof. Since we now assume that $\partial_{ss}\rho(0,\theta) = 0$, the computation above give in particular

$$\partial_{ss}N(0,\theta) = 0, \quad \partial_{ss}D(0,\theta) = 0.$$

Next, we define

$$a(s,\theta) = ((\partial_s n)d - n(\partial_s d))(s,\theta) \quad \text{and} \quad b(s,\theta) = d^2(s,\theta),$$

and we have that

$$\partial_{sss}k(s,\theta) = \left(\frac{(\partial_s a)b - a(\partial_s b)}{b^2}\right)(s,\theta), \tag{B.4}$$

where

$$(\partial_s a)(s,\theta) = ((\partial_{ss}n)d - n(\partial_{ss}d))(s,\theta) \quad \text{and} \quad (\partial_s b)(s,\theta) = 2(d\,\partial_s d)(s,\theta),$$

with

$$(\partial_{ss}n)(s,\theta) = ((\partial_{sss}N)D - N(\partial_{sss}D) + (\partial_{ss}N)\partial_s D - (\partial_s N)(\partial_{ss}D))(s,\theta),$$
$$(\partial_{ss}d)(s,\theta) = 2((\partial_s D)^2 + D\,\partial_{ss}D)(s,\theta)$$

in which

$$(\partial_{sss}N)(s,\theta) = (6\partial_s\rho\,\partial_{ss}\rho + (2R_0+2\rho-\partial_{\theta\theta}\rho)\partial_{sss}\rho + 12\partial_{s\theta}\rho\,\partial_{ss\theta}\rho$$
$$+ 4\partial_\theta\rho\partial_{sss\theta}\rho - (R_0+\rho)\partial_{sss\theta\theta}\rho - 3\partial_{ss}\rho\partial_{s\theta\theta}\rho - 3\partial_s\rho\partial_{ss\theta\theta}\rho)(s,\theta)$$

and

$$\begin{aligned}(\partial_{sss}D)(s,\theta) = & -3((R_0+\rho)^2+(\partial_\theta\rho)^2)^{-3/2}((R_0+\rho)\partial_s\rho+\partial_\theta\rho\partial_{s\theta}\rho)^3 \\ & + 9((R_0+\rho)^2+(\partial_\theta\rho)^2)^{-1/2}((R_0+\rho)\partial_s\rho+\partial_\theta\rho\partial_{s\theta}\rho)(\partial_\theta\rho\partial_{ss\theta}\rho+(\partial_s\rho)^2) \\ & + 9((R_0+\rho)^2+(\partial_\theta\rho)^2)^{-1/2}((R_0+\rho)\partial_s\rho+\partial_\theta\rho\partial_{s\theta}\rho)\left((R_0+\rho)\partial_{ss}\rho+(\partial_{s\theta}\rho)^2\right) \\ & + 3\sqrt{(R_0+\rho)^2+(\partial_\theta\rho)^2}(3\partial_s\rho\partial_{ss}\rho+(R_0+\rho)\partial_{sss}\rho) \\ & + 3\sqrt{(R_0+\rho)^2+(\partial_\theta\rho)^2}(3\partial_{s\theta}\rho\partial_{ss\theta}\rho+\partial_\theta\rho\partial_{sss\theta}\rho)(s,\theta).\end{aligned}$$

Using that $\rho(0,\theta) = \partial_s\rho(0,\theta) = \partial_{ss}\rho(0,\theta) = 0$, we get:

$$\partial_{sss}N(0,\theta) = 2R_0\partial_{sss}\rho(0,\theta) - R_0\partial_{sss\theta\theta}\rho(0,\theta),$$

and

$$\partial_{sss}D(0,\theta) = 3R_0^2\partial_{sss}\rho(0,\theta).$$

Since $\partial_sD(0,\theta) = \partial_{ss}D(0,\theta) = 0$, $\partial_sd(0,\theta) = 0$, $\partial_sb(0,\theta) = 0$, we deduce that

$$\begin{aligned}R_0^{12}\partial_{sss}k(0,\theta) = \partial_sa(0,\theta) &= R_0^6\partial_{ss}n(0,\theta) \\ &= R_0^6[(\partial_{sss}N)D - N(\partial_{sss}D)](0,\theta) \\ &= R_0^{10}[-\partial_{sss}\rho(0,\theta) - \partial_{sss\theta\theta}\rho(0,\theta)]\end{aligned}$$

and the result follows. □

C Computation of $\partial_s z$, $\partial_{ss}z$ and $\partial_{sss}z$

Recall that $z(s,\theta)$ is defined by (4.19) for $s\in I$ and $\theta\in[-\pi,+\pi]$ and that $c(s)$ satisfies

$$c(0) = 0, c'(0) = 1.$$

Lemma C.1. *Assume $\rho(0,\theta) = \partial_s\rho(0,\theta) = 0$ for all $\theta\in[-\pi,+\pi]$. Then for all $\theta\in[-\pi,\pi]$ we have*

$$\partial_sz(0,\theta) = \cos\theta, \quad \text{and} \quad \partial_{ss}z(0,\theta) = c''(0)\cos\theta$$

Proof. Define the functions

$$\mathcal{N}(s,\theta) = (R_0+\rho(s,\theta))\cos\theta + \partial_\theta\rho(s,\theta)\sin\theta,$$
$$\mathcal{D}(s,\theta) = \sqrt{(R_0+\rho(s,\theta))^2+(\partial_\theta\rho(s,\theta))^2},$$

so that

$$z(s,\theta) = c(s)\frac{\mathcal{N}(s,\theta)}{\mathcal{D}(s,\theta)} \quad \text{and} \quad \partial_s z(s,\theta) = c'(s)\frac{\mathcal{N}(s,\theta)}{\mathcal{D}(s,\theta)} + c(s)\frac{\nu(s,\theta)}{\delta(s,\theta)}, \tag{C.1}$$

where

$$\nu(s,\theta) = ((\partial_s\mathcal{N})D - \mathcal{N}(\partial_s\mathcal{D}))(s,\theta) \quad \text{and} \quad \delta(s,\theta) = \mathcal{D}^2(s,\theta).$$

We have $\mathcal{N}(0,\theta) = R_0\cos\theta$ and $\mathcal{D}(0,\theta) = R_0$ and since $c(0) = 0$ and $c'(0) = 1$, we obtain $\partial_s z(0,\theta) = \frac{\mathcal{N}(0,\theta)}{\mathcal{D}(0,\theta)} = \cos\theta$ which is the first part of the lemma.

Differentiating (C.1) with respect to s we obtain

$$\partial_{ss}z(s,\theta) = c''(s)\frac{\mathcal{N}(s,\theta)}{\mathcal{D}(s,\theta)} + 2c'(s)\frac{\nu(s,\theta)}{\delta(s,\theta)} + c(s)\frac{h(s,\theta)}{g(s,\theta)}, \tag{C.2}$$

where

$$h(s,\theta) = ((\partial_s\nu)\delta - \nu(\partial_s\delta))(s,\theta) \quad \text{and} \quad g(s,\theta) = \delta^2(s,\theta).$$

Recalling that $c(0) = 0$ and $c'(0) = 1$, we find

$$\partial_{ss}z(0,\theta) = c''(0)\frac{\mathcal{N}(0,\theta)}{\mathcal{D}(0,\theta)} + 2\frac{\nu(0,\theta)}{\delta(0,\theta)}.$$

Next, we have:

$$\partial_s\mathcal{N}(s,\theta) = \partial_s\rho(s,\theta)\cos\theta + \partial_{s\theta}\rho(s,\theta)\sin\theta,$$

and

$$\partial_s\mathcal{D}(s,\theta) = \frac{(R_0 + \rho(s,\theta)\partial_s\rho(s,\theta) + \partial_\theta\rho(s,\theta)\partial_{s\theta}\rho(s,\theta)}{\sqrt{(R_0+\rho(s,\theta))^2 + \partial_\theta\rho(s,\theta)^2}}.$$

In particular, we get $\partial_s\mathcal{N}(0,\theta) = 0$, $\partial_s\mathcal{D}(0,\theta) = 0$, hence

$$\nu(0,\theta) = 0, \qquad \delta(0,\theta) = R_0^2.$$

We deduce:

$$\partial_{ss}z(0,\theta) = c''(0)\frac{\mathcal{N}(0,\theta)}{\mathcal{D}(0,\theta)} + 2\frac{\nu(0,\theta)}{\delta(0,\theta)} = c''(0)\cos\theta.$$

□

Lemma C.2. *Assume $\rho(0,\theta) = \partial_s\rho(0,\theta) = \partial_{ss}\rho(0,\theta) = 0$ for all $\theta \in [-\pi,+\pi]$. Then for all $\theta \in [-\pi,\pi]$ we have*

$$\partial_{sss}z(0,\theta) = c'''(0)\cos\theta$$

Proof. We must go one step further. Differentiating (C.2) with respect to s we find

$$\partial_{sss}z(s,\theta) = c'''(s)\frac{\mathcal{N}(s,\theta)}{\mathcal{D}(s,\theta)} + 3\,c''(s)\frac{\nu(s,\theta)}{\delta(s,\theta)} + 3\,c'(s)\frac{h(s,\theta)}{g(s,\theta)} + c(s)\,\partial_s\left(\frac{h(s,\theta)}{g(s,\theta)}\right).$$

Since $c(0) = 0$, $c'(0) = 1$ and $\nu(0,\theta) = 0$ (see proof of the previous lemma) we obtain

$$\partial_{sss} z(0,\theta) = c'''(0) \frac{\mathcal{N}(0,\theta)}{\mathcal{D}(0,\theta)} + 3 \frac{h(0,\theta)}{g(0,\theta)}$$

We have

$$(\partial_{ss}\mathcal{N})(s,\theta) = \partial_{ss}\rho(s,\theta)\cos\theta + \partial_{ss\theta}\rho(s,\theta)\sin\theta,$$

and

$$\begin{aligned}
&(\partial_{ss}\mathcal{D})(s,\theta) = \\
&-((R_0+\rho)^2 + (\partial_\theta\rho)^2)^{3/2}((R_0+\rho)\partial_s\rho + \partial_\theta\rho\partial_{s\theta}\rho)((R_0+\rho)\partial_s\rho + \partial_\theta\rho\partial_{s\theta}\rho) \\
&+((R_0+\rho)^2 + (\partial_\theta\rho)^2)^{-1/2}(R_0\partial_{ss}\rho + (\partial_s\rho)^2 + \rho\partial_{ss}\rho + (\partial_{s\theta}\rho)^2 + \partial_\theta\rho\partial_{ss\theta}\rho)(s,\theta)
\end{aligned}$$

so $\partial_{ss}\mathcal{N}(0,\theta) = 0$ and $\partial_{ss}\mathcal{D}(0,\theta) = 0$ and so

$$(\partial_s\nu)(0,\theta) = ((\partial_{ss}\mathcal{N})\mathcal{D} - \mathcal{N}(\partial_{ss}D))(0,\theta) = 0.$$

In turns this gives

$$h(0,\theta) = ((\partial_s\nu)\delta - \nu(\partial_s\delta))(0,\theta) = 0.$$

Together with the fact that $\mathcal{N}(0,\theta) = R_0\cos\theta$ and $\mathcal{D}(0,\theta) = R_0$, this implies

$$\partial_{sss} z(0,\theta) = c'''(0) \frac{\mathcal{N}(0,\theta)}{\mathcal{D}(0,\theta)} + 3 \frac{h(0,\theta)}{g(0,\theta)} = c'''(0)\cos\theta.$$

□

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