Lecture 1: Abstract Vector Spaces

The Definition of a Field

This page comes from Chapter 1, page 8 of the text. Examples of fields are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ and the rational numbers $\mathbb{Q}$. There are also finite fields, for example, $\mathbb{Z}/p$, p a prime.

Definition

A **field** F is a set (also denoted F) equipped with two binary operations, addition $+$ and multiplication $\cdot$ satisfying the following axioms

1. $x + y = y + x$ and $x \cdot y = y \cdot x$ (the commutative laws)
2. $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (the associative laws)
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ (the distributive law)
4. There exists an element 0 in F such that $x + 0 = x$ for all $x \in F$.
5. For each $x \in F$ there exists an element $-x$ such that $x + (-x) = 0$.
6. There exists an element 1 in F such that $x \cdot 1 = x$ for all $x \in F$.
7. For each $x \in F$ with $x \neq 0$ there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.

We will usually write xy instead of $x \cdot y$.

Vector Space over a Field F

We now skip to Chapter 2.

Definition

A **vector space over** F is a triple $(V, +, \cdot)$ where,

1. V is a set,
2. $+$ is a binary operator that assigns to any pair $v_1, v_2 \in V$ a new element $v_1 + v_2 \in V$,
3. $\cdot$ is a binary operation that assigns to any pair $c \in F$ and $v \in V$ a new vector $c \cdot v \in V$.

The operation $+$ satisfies 5 axioms.

Axioms for Addition $+$

A1 Commutativity

$$u + v = v + u.$$

A2 Associativity

$$(u + v) + w = u + (v + w).$$

A3 Existence of the zero vector
There exists a unique element 0 of V such that

$$v + 0 = v, \text{ for all } v \in V.$$

A4 Existence of an additive inverse
For each $v \in V$, there exists a vector $-v$ such that

$$v + (-v) = 0.$$

We will abbreviate $u + (-v)$ for $u - v$, so we have defined subtraction.

Axioms for scalar multiplication $\cdot$

S1 Associativity

$$c_1 \cdot (c_2 v) = (c_1 c_2) v.$$

S2 Distributivity (1^{st} version)

$$(c_1 + c_2) \cdot v = c_1 \cdot v + c_2 \cdot v.$$

S3 Distributivity (2^{nd} version)

$$c \cdot (v_1 + v_2) = c \cdot v_1 + c \cdot v_2.$$

S4

$$1 \cdot v = v.$$

Vector Space Axioms

We will call the axioms $A1$, $A2$, $A3$, $A4$ and $S1$, $S2$, $S3$, $S4$ the vector space axioms.

We will prove shortly that

$$0 \cdot v = 0,$$

and

$$(-1)v = -v.$$

The Main Examples

Eg. I $\mathbb{R}^n$

As a set $\mathbb{R}^n$ is the set of ordered n-tuples

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}.$$

We have to define the operator $+$ and $\cdot$.

Addition

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n).$$

Scalar Multiplication

$$c \cdot (x_1, \ldots, x_n) := (cx_1, \ldots, cx_n).$$

Theorem

This works, that is, the eight vector space axioms are satisfied.

Define vectors $(e_1, e_2, \ldots, e_n) \in \mathbb{R}^n$ by $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, etc.

The Main Examples

Eg. II The space of real-valued functions on a set X

Let X be a set and $\mathcal{F}_{\mathbb{R}}(X)$ be the set of real-valued function on the set X. We define $+$ and $\cdot$ by

$$\begin{aligned}(f+g)(x) &:= f(x)+g(x)\\(c\cdot f)(x) &:= cf(x).\end{aligned}$$

<u>Exercise</u>
Show that Example II includes Example I.
Hint: Take X to be the n-element set $\{1, 2, \ldots, n\}$.

Properties of $+$ and $\cdot$ that can be deduced from the axioms.

Theorem (3.5)

Let V be a vector space over F. Then the following statements hold

(1) *Cancellation*

$$u + w = v + w \Longrightarrow u = v.$$

(2) *The equation $u + x = v$ has unique solution*

$$x = v - u.$$

(3) $0 \cdot u = 0$.

(4) $(-1) \cdot u = -u$.

(5) $c_1 \cdot u = c_2 \cdot u$ *and* $u \neq 0 \Longrightarrow c_1 = c_2$

Proof.

(1) Add $-w$ to each side.

(2) Add $-u$ to each side.

(3) This one is tricky!
Let 0 be the zero element in F (!! not the zero element in V). Then

$$\begin{aligned} 0+0 &= 0 \\ (0+0)\cdot u = 0\cdot u \\ 0\cdot u + 0\cdot u = 0\cdot u. \end{aligned}$$

Subtract the vector $0\cdot u$ from each side to get

$$0\cdot u = 0.$$

Proof (continued).

(4) We want to show

$$u + (-1)\cdot u = 0 \qquad (*)$$

From S4, $(1\cdot)u = u$, so

$$\begin{aligned} LHS(*) &= (1)\cdot u + (-1)u = (1 + (-1))u \\ 0\cdot u &= 0 \text{ from } (3). \end{aligned}$$

Proof (continued).

(5) Suppose $u \neq 0$ and $c_1 \cdot u = c_2 \cdot u$. Hence

$$(c_1 - c_2) \cdot u = 0 \quad (**).$$

We want to prove $c_1 - c_2 = 0$ in F. Suppose not. Then $(c_1 - c_2)^{-1} \in F$ exists. Multiply both sides of $(**)$ by $(c_1 - c_2)^{-1}$ to get $(c_1 - c_2)^{-1} \cdot ((c_1 - c_2) \cdot u) = (c_1 - c_2)^{-1} \cdot 0 = 0$.

$$LHS = \big((c_1 - c_2)^{-1}(c_1 - c_2) \cdot u\big) = 1 \cdot u = u.$$

But RHS of $(**)$ equals 0 and hence $u = 0$, contradicting our assumption that $u \neq 0$. Hence, our assumption that $c_1 - c_2 \neq 0$ has led to a contradiction. Hence $c_1 - c_2 = 0$ and $c_1 = c_2$. □