Lecture 3: Bases

(We will consider only vector spaces that have finite spanning sets.)

Basics on Bases

Definition

Let V be a vector space and $\{v_1, \ldots, v_n\} \subset V$. Then $\{v_1, \ldots, v_n\}$ is a **basis** for V if

(i) $\{v_1, \ldots, v_n\}$ spans V.

(ii) $\{v_1, \ldots, v_n\}$ is linearly independent.

Today we will prove two of the main foundational theorems in linear algebra.

First Main Theorem (Text, Theorem 5.3)

Every vector space V has a basis (in fact, many bases).

Second Main Theorem (Text, Theorem 7.2)

Any two bases of V have the same cardinality.

Usefulness of Basis

But first–why are bases useful?

Proposition

Suppose $\mathscr{B} = \{v_1, \ldots, v_n\}$ *is a basis for* V. *Let* $v \in V$. *Then there exist unique scalars* c_1, c_2, ..., c_n *such that*

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

The scalars c_1, c_2, ..., c_n *are said to be the coordinates of* v *relative to the basis* $\{v_1, \ldots, v_n\}$.

Usefulness of Bases

Proof. The c_i's exist because the v_i's span V. We will prove that they are unique. Suppose V has two sets of coordinates relative to $\{v_1, \ldots, v_n\}$, i.e.,

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

and

$$v = c_1' v_1 + c_2' v_2 + \ldots + c_n' v_n.$$

Then $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = c_1' v_1 + c_2' v_2 + \ldots + c_n' v_n$, so

$$(c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \ldots + (c_n - c_n')v_n = 0$$

so $c_i - c_i' s = 0$. □

Future example

In Lecture 7, we will introduce the notation $[v]_{\mathscr{B}}$ for the coordinates of a vector v relative to a basis $\mathscr{B}$.

Problem (to be solved in Lecture 7)
Suppose $\mathscr{A} = \{a_1, \ldots, a_n\}$ and $\mathscr{B} = \{b_1, \ldots, b_n\}$ are both bases for V. Let $v \in V$. How are the coordinates $[v]_{\mathscr{A}}$ of V relative to $\mathscr{A}$ related to the coordinates $[v]_{\mathscr{B}}$ of v relative to $\mathscr{B}$?

First Main Theorem

We will now prove the First Main Theorem.

First Main Theorem

Every vector space which has a finite spanning set has a basis.

Proof. First we have to take care of the zero vector space $\{0\}$. The empty set is a basis for $\{0\}$. (We will agree that the 0-vector is a combination of the vectors in the empty set.)
Now let V be a non-zero vector space which has a finite spanning set–say with m elements, List the cardinalities of all spanning sets with at most m elements. This is a subset of $\{1, 2, \ldots, m\}$ and has a smallest element, n. Hence there is a set of vectors $\{v_1, \ldots, v_n\} \subset V$ such that

(1) $\{v_1, \ldots, v_n\}$ spans V

(2) No subset of $\{v_1, \ldots, v_n\}$ spans V.

We claim that $\{v_1, \ldots, v_n\}$ is linearly independent and hence a basis. If not, one of the vectors v_i is a combination of the rest and $\{v_1, \ldots, \hat{v}_i, \ldots, v_n\}$ spans V. But $\#\{v_1, \ldots, \hat{v}_i, \ldots, v_n\} =$ n-1. Contradiction. □

We will now prove the **Second Main Theorem:** Any two bases have the same cardinality (same number of elements).

Definition

The number of vectors in a basis for V is said to be the *dimension* of V.

The Second Main Theorem will follow from the next theorem.

Theorem

Suppose V is a vector space and $\{u_1, \ldots, u_m\}$ is a spanning set for V. Then any subset of V with more than m elements is linearly dependent.

Proof. I will prove this theorem using a theorem from linear equations.

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

$m < n \Longrightarrow$ the system has a nonzero solution.

Now let $v_1, \ldots, v_n$ be an $n-$element set with $n > m$. We want to find $x_1, \ldots, x_n$ not all zero so that $x_1 v_1 + \ldots + x_n v_n = 0$.
Write

$$\begin{aligned} v_1 = a_{11}u_1 + a_{21}u_2 + \ldots + a_{m1}u_m &= 0 \\ v_2 = a_{12}u_1 + a_{22}u_2 + \ldots + a_{m2}u_m &= 0 \\ &\vdots \\ v_n = a_{1n}u_1 + a_{2n}u_2 + \ldots + a_{mn}u_m &= 0 \end{aligned}$$

Then

$$\begin{aligned} x_1 v_1 + \ldots + x_n v_n &= x_1 a_{11} u_1 + x_1 a_{21} u_2 + \ldots + x_1 a_{m1} u_m \\ &+ x_2 a_{12} u_1 + x_2 a_{22} u_2 + \ldots + x_2 a_{m2} u_m \\ &+ \vdots \\ &+ x_n a_{1n} u_1 + x_n a_{2n} u_2 + \ldots + x_n a_{mn} u_m \\ &= (a_{11}x_1 + \ldots + a_{1n}x_n)u_1 + (a_{21}x_1 + \ldots + a_{2n}x_n)u_2 \\ &+ \ldots + (a_{m1}x_1 + \ldots + a_{mn}x_n)u_m \end{aligned}$$

$x_1 v_1 + \ldots + x_n v_n = 0 \iff x_1, \ldots, x_n$ satisfy

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

But $n > m$ so there are more unknowns than equations. Hence there is a nonzero solution. □

Corollary

The cardinality of any linearly independent set is always less than or equal to the cardinality of any spanning set.

Second Main Theorem

Second Main Theorem

Suppose $\{w_1, \ldots, w_m\}$ *and* $\{v_1, \ldots, v_n\}$ *are both bases of* v. *Then* $n = m$.

Proof. Since $\{w_1, \ldots, w_m\}$ spans and $\{v_1, \ldots, v_n\}$ is linearly independent we have $n \leq m$. But $\{v_1, \ldots, v_n\}$ spans and $\{w_1, \ldots, w_m\}$ is linearly independent, hence $m \leq n$. $\square$

Example: $\dim \mathbb{R}^n = n$ because $\mathscr{E} = \{e_1, e_2, \ldots, e_n\}$, where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, ..., $e_n = (0, 0, \ldots, 1)$), is a basis.

Bases for Infinite dimensional vector spaces

We need the definitions of a spanning set and an independent set in an infinite dimensional vector space.

Definition

An infinite subset S of an infinite dimensional vector space V spans V if every vector $v \in V$ is a finite linear combination of elements of S.

Definition

An infinite subset S of an infinite dimensional vector space V is an independent set if no nontrivial linear combination of a finite number of elements of S is equal to zero.

The next problem will be on Homework 1.

Problem

Show that the infinite set $\{1, x^2, x^3, \cdots, x^n, \cdots\}$ is a basis for the vector space of polynomials in one variable x.

Infinite dimensional vector spaces have bases

It is in fact true that **every** vector space has a basis but it is harder to prove. You need to prove

(1) Every vector space has a maximal linearly independent set $\mathscr{B}$ (this means there is no linearly independent set containing $\mathscr{B}$).

(2) Any maximal linearly independent set $\mathscr{B}$ spans and hence is a basis.

You should be able to prove (2). The idea behind (1) is simple but the actual proof of (1) involves subtleties from set theory/logic (Zorn's Lemma).