Lecture 5: More on Linear Transformations

Today, we tidy up some odds and ends.

Theorem (Text, Theorem 13.1)

Let $\mathscr{B} = \{b_1, \ldots, b_n\}$ *be a basis for* V*. Let* $w_1, \ldots, w_n$ *be arbitrary vectors in* W*. Then there exists a unique* $T \in \mathrm{Hom}(V, W)$ *with*

$$T(b_i) = w_i, \quad 1 \le i \le n.$$

Proof. Uniqueness is clear.

Existence: Define $T(v)$ for $v \in V$ as follows. Write $v = x_1 b_1 + \ldots x_n b_n$ and "define"

$$T(v) = \sum_{i=1}^{n} x_i w_i.$$

Is T well-defined?
Yes. Since $\mathscr{B}$ is a basis, the x_i's are uniquely determined. Also, note that T is linear. □

The Range and Null Space (Kernel) of a Linear Transformation

There are two useful subsets.

Definition

The **Range** or **Image** of T, denoted $T(V)$ or $R(T)$ is defined as

$$T(V) := \{T(v) : v \in V\} \subset W.$$

Lemma: $T(V)$ is a subspace of W.
Proof. Use the fact that T is linear.

Definition

The **Nullspace** of T, denoted $N(T)$ is defined as

$$N(T) := \{v \in V : T(v) = 0\} \subset V.$$

We will often use the word kernel of T, denoted $ker(T)$, instead of the nullspace of T. We leave the proof of the next lemma to you.

Lemma: N(T) is a subspace of V.

A dimension formula

Theorem (Text, Theorem 13.9)

Let $T \in \mathrm{Hom}(V, W)$. *Then*

$$\dim T(V) + \dim N(T) = \dim V.$$

To prove this, first we need the following proposition (we assume V is finite-dimensional).:

Proposition

Any linearly independent set $\mathcal{S} = \{v_1, \ldots, v_k\} \subset V$ *can be completed to a basis* $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ *for* V.

Proof.

If $\mathcal{S}$ spans then it is a basis. If not there is a vector u not in the span of $\mathcal{S}$. Then $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n, u\}$ is still an independent set (prove this). Continue. Since we are assuming V is finite dimensional,say m the process must stop after $m - n$ steps. □

Proof of Theorem 13.9. Assume $\dim V = n$ and $\dim N(T) = k$. Choose a basis $\{b_1, \ldots, b_k\}$ for $N(T)$ and complete it to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ for V. It sufficed to prove

Claim: $\{T(b_{k+1}), \ldots, T(b_n)\}$ is a basis for $R(T)$.

Spanning set: Clear.

Independent set: Suppose $x_{k+1}T(b_{k+1}) + \ldots + x_nT(b_n) = 0$. Then $x_{k+1}b_{k+1} + \ldots + x_nb_n \in N(T)$ and hence $x_{k+1}b_{k+1} + \ldots + x_nb_n = x_1b_1 + \ldots + x_kb_k$.

Thus

$$x_1b_1 + \ldots + x_kb_k - x_{k+1}b_{k+1} - \ldots - x_nb_n = 0$$

But $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ is a basis, so all the coefficients x_i, $1 \leq i \leq n$ are zero. Hence $x_{k+1}, \ldots, x_n$ are zero. □

A Consequence of Theorem 13.9

Our next goal is to prove the following proposition.

Proposition

Let $T \in \operatorname{End}(V)$ (so $V = W$). Then T is 1:1 $\Longleftrightarrow$ T is onto.

We will need the next lemma. This lemma is extraordinarily useful.

Lemma

Suppose $T : V \longrightarrow W$. Then T is 1:1 $\Longleftrightarrow$ $N(T) = \{0\}$.

Proof. ($\Longrightarrow$) Suppose $T(v) = 0$. Then since $T(0) = 0$, we have $v = 0$, hence if $v \in N(T)$ then $v = 0$.
($\Longleftarrow$) Suppose $T(v_1) = T(v_2)$. Then since T is linear $T(v_1 - v_2) = 0$, hence $v_1 - v_2 \in N(T)$ and thus $v_1 - v_2 = 0$. Finally, $v_1 = v_2$. □

Proof of the previous Proposition

Proof. We use $\dim V = \dim R(T) + \dim N(T)$.
($\Longrightarrow$) T is 1:1 so $N(T) = 0$, hence $\dim R(T) \dim V$, thus $R(T) = V$ (since $R(T) \subset V$).
($\Longleftarrow$) T is onto, so $R(T) = V$ and $\dim R(T) = \dim V$. Thus $\dim N(T) = 0$ and so $N(T) = \{0\}$.

Warning: This is not true if $W \neq V$.

Now do problem pg. 108 # 10.