Lecture 26

Calculation of the Jordan Normal Form and the Jordan basis (equivalently P^{-1}) — three examples

In what follows I will compute the Jordan normal forms for three examples

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

You can see the computation for the last one (and some other useful things about Jordan normal form) online. Put "computing Jordan normal form" into the Google search box and go to "Math Doctor Bob".

For these cases of small matrices it is easy to find the Jordan normal form (but not P^{-1}) using the following theorem.

Theorem 1

(1) For 2 by 2 and 3 by 3 matrices the Jordan normal form is determined by the multiplicities of the eigenvalues

(2) For 4 by 4 matrices the Jordan normal form is determined by the multiplicities of the eigenvalues and the minimal polynomial $m(x)$ (so the sizes of the biggest λ-block for each λ).

Example 1

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

Remark This example has a special property. The characteristic polynomial has only one root $x=1$ but the eigenvalue 1 has multiplicity 2. So we have to find only one generalized eigenvector.

Computation of $J(A)$

$$xI - A = \begin{pmatrix} x+1 & 1 & 0 \\ 0 & x+1 & 2 \\ 0 & 0 & x+1 \end{pmatrix} \quad \text{so } h(x) = (x+1)^3$$

so the Jordan normal form is either of the form

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In the first case the minimal polynomial is $m(x) = (x+1)^3$ and in the second $m(x) = (x+1)^2$

Recall that $E_\lambda(A) = \ker(A-\lambda I) \subset V$ is the space of all eigenvectors of A belonging to the eigenvalue λ so it is important to write down a basis for this space, that is find the general solution to

$$(A-\lambda I)v = 0 \qquad (*)$$

We now do this for the above case

so $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$ and $\lambda = -1$

Hence

$$(A-(-1)I) = A+I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

and putting $v = (x, y, z)$ we get

$$(A+I)v = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -2z \\ 0 \end{pmatrix}$$

so (x, y, z) satisfies $(*)$ $\iff$ $y=0$ and $z=0$

So the general solution to $(*)$ is $\{(x, 0, 0)\} = E_{-1}(A)$ and a basis for the solution space is the first standard basis vector $e_1 = (1, 0, 0)$

So e_1 is an eigenvector belonging to $\lambda = -1$.

So the eigenvalue -1 has multiplicity $\underline{1}$ so there is only one Jordan block. So the Jordan normal form $J(A)$ for A is given by

$$J(A) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

<u>Remark</u> We have computed the Jordan normal form of A without computing P (or any generalized eigenvectors belonging to -1).

<u>Computing the matrix P (equivalently a Jordan basis)</u>

We need to compute a matrix P such that

$$P A P^{-1} = J(A)$$

We know the first column of P is the eigenvector $\underline{e_1}$ belonging to $\lambda = -1$ so

$$P^{-1} = \begin{pmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{pmatrix}$$

Because there is only one eigenvector for A (up to scalar multiples) there must be a Jordan basis for V of the form $\mathcal{B} = \{ v_1 = e_1, v_2, v_3 \}$

where v_2 is a level two generalized eigenvector (so $(A-\lambda I)^2 v_2 = 0$) associated to e_1 so, that is,

$$(A-\lambda I)v_2 = v_1 \quad (**)$$

and v_3 is a level three generalized eigenvector (so $(A-\lambda I)^3 v_3 = 0$) associated to v_2 so

$$(A-\lambda I)v_3 = v_2 \quad (***)$$

Once we have solved $(**)$ and $(***)$ then

$$P^{-1} = \begin{array}{ccc} e_1 & v_2 & v_3 \\ \end{array}$$
$$\left(\begin{array}{ccc} \downarrow & \downarrow & \downarrow \end{array} \right)$$

Remark: We don't know what the equation $(**)$ is until we have found v_1; that is, until we have found an eigenvector v_1 belonging to λ (so $v_1 = e_1$ in this case) and we don't know the equation $(***)$ until we have solved $(**)$ to get v_2.

Solving the equation (**) to find the second basis vector

We now compute v_2, the second vector in the Jordan basis, by finding a solution to the equation (**)

$$(A-\lambda I)v_2 = e_1 \quad (**)$$

Putting $v_2 = (x, y, z)$ and noting

$$A-\lambda I = A+I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} -y \\ -2z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (**)$$

So $y=-1$, $z=0$ and x is not determined and the general solution to (**) is $(x, -1, 0)$ so again one dimensional. We can choose any solution for v_2. We choose the simplest solution $x=0$ so $v_2 = (0, -1, 0)$

So we have found the first two columns of P^{-1}

$$P^{-1} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & -1 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}$$

Solving the equation (xxx) to find the third basis vector

Now we have chosen $v_2 = (0, -1, 0)$ we can write down the equation (~~xxx~~)

$$(A - \lambda I) v_3 = v_2 \qquad (xxx)$$

that is

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} -y \\ -2z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \qquad (xxx)$$

The general solution to (xxx) is then

$(x, 0, \frac{1}{2})$ and the simplest solution is $(0, 0, \frac{1}{2})$

Hence, $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ and so $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Check the answer

We need to verify $P A P^{-1} = B = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & & -1 \end{pmatrix}$

$$LHS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Yes

Problem What is the minimal polynomial $m_A(x)$ for this case?

Example 2

(an unpleasant surprise concerning solving (**) for the first generalized eigenvector)

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Then $h_A(x) = \det \begin{pmatrix} x-2 & -1 & -1 \\ 1 & x & 1 \\ 0 & 0 & x-1 \end{pmatrix}$

$$= [(x-2)x + 1](x-1) = (x^2-2x+1)(x-1)$$
$$= (x-1)^3$$

So the only eigenvalue of A is $\lambda = 1$

Now we have to compute the multiplicity of $\lambda=1$ as an eigenvalue so we need the general solution to

$$(A - \lambda I)v = 0 \quad \text{so} \quad (A - I)v = 0$$

Substituting for A and λ we get

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{aligned} x+y+z &= 0 \\ -x-y-z &= 0 \\ 0 &= 0 \end{aligned} \qquad (*)$$

There is only one independent equation here. So the eigenspace $E_1(A)$ belonging to the eigenvalue $\lambda=1$ is given by

$$E_1(A) = \{(x,y,z) : x+y+z=0\}$$

Hence $E_1(A)$ is a plane. We choose the basis $(1,-1,0)$ and $(1,0,-1)$ for the plane $E_1(A)$.

Now to get a Jordan basis for $B=\{(1,-1,0),(1,0-1),v_2\}$ we need to find a level 2 generalized eigenvector v_2 so that

$$(A-\lambda I)v_2 = (1,0,-1) \qquad (**)$$

or

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

or

$$\begin{aligned} x+y+z &= 1 \\ -x-y-z &= -1 \\ 0 &= -1 \leftarrow \text{no} \end{aligned}$$

Clearly there is no solution to this system.

The problem is that $(1,0,-1) \in E_1(A)$ was the wrong choice of eigenvector for the right-hand side of (**). $E_1(A)$ is a plane so there are lots of choices. We want to choose a vector (a,b,c) in $E_1(A)$ so we can solve (**)

$$x+y+z=a$$
$$-x-y-z=b$$
$$0=c$$

So we that we need to have $b=-a$ and $c=0$. So the vector $(1,-1,0)$ works. So our revised version of (**) (solving for v_2) is

$$x+y+z=1$$
$$-x-y-z=-1$$
$$0=0$$

These three equations are equivalent to the single equation

$$x+y+z=1 \qquad (**)$$

So once again there is a plane of solutions and we can pick any vector in that plane for v_2. The simplest solution is $x=1, y=0, z=0$ so $v_2 = e_1 = (1,0,0)$. We obtain the basis $\mathcal{B} = \{(1,-1,0), (1,0,-1), (1,0,0)\}$ and $P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. But this doesn't work.

By definition of a Jordan basis
$(A-\lambda I)$ last basis vector $=$ next-to-last basis vector.
But in our case $(A-\lambda I)e_1 = (1,-1,0)$.
This has as a consequence if we write out B the matrix for A using the basis $\mathcal{B}$ we will get (since $Ae_1 = (2,-1,0) = (1,-1,0) + e_1$)

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \text{not in Jordan normal form}$$

To get a Jordan basis we have to exchange the first and second basis vectors and redefine $\mathcal{B}$ by

$$\mathcal{B} = \{(1,0,-1), (1,-1,0), \overset{A-I}{\curvearrowleft}(1,0,0)\}.$$

Now $\mathcal{B}$ is a Jordan basis and the new B will satisfy $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

We now have $P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

Let's check that this P does the job

Lemma

$$P \; A \; P = J(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof $P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ so $P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Then

$$PAP^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \square$$

Remark I will leave it to you to invert P^{-1} to get P. It isn't easy.

Example 3 (from Math Doctor Bob online)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Step 1: Compute $h_A(x)$

$$h_A(x) = \det\left(\begin{pmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & -1 & 0 \\ 0 & 0 & x-2 & 0 \\ -1 & 0 & 0 & x-2 \end{pmatrix}\right)$$

expand using top row

$$= (x-2)^4$$

Step 2. Compute the multiplicity $m(2)$ of the eigenvalue $\lambda = 2$

$$m(2) = \dim\left(\ker(A-2I)\right)$$

But

$$A-2I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

so

$$(A-2I)\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ 0 \\ x \end{pmatrix} \quad \text{so} \quad (A-2I)\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow$ $x = 0$ and $z = 0$

Hence $E_2(A) = \ker(A-2I)$
$$= \{(0, y, 0, t)\}$$

Hence the eigenvalue $\lambda = 2$ has multiplicity two so there are two Jordan blocks of size 3 by 3 and 1 by 1 or 2 by 2 and 2 by 2.

Step 3 Find $J(A)$ by computing $m_A(x)$

So we need to find the size of the biggest block. So we need to know the degree of the minimal polynomial $m_A(x)$. Since the characteristic polynomial $h_A(x) = (x-2)^4$ and $m_A(x) \mid h_A(x)$ we have either $m_A(x) = (x-2)^2$ or $m_A(x) = (x-2)^3$ depending on whether the biggest Jordan block has size 2 or 3. In case $m_A(x) = (x-2)^2$ then $(A-2I)^2 = 0$ and if $m_A(x) = (x-2)^3$ then $(A-2I)^2 \neq 0$ but $(A-2I)^3 = 0$.

So it comes down to checking whether or not $(A-2I)^2 = 0$

But $A-2I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and so

$$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 0$$

Hence we have found the Jordan normal form for A namely

$$J(A) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Step 4 <u>Find P^{-1} or equivalently a Jordan basis $\mathcal{B}$ for A</u>

We have seen $E_2(A) = \{(0, y, 0, t)\}$ so a basis for $E_2(A)$ is $\{e_2, e_4\}$ where $e_2 = (0,1,0,0)$, $e_4 = (0,0,0,1)$

In order to find a Jordan basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$. So we need to find a generalized eigenvector u belonging to e_2

so $(A-2I)u = e_2$ (**)

and by a generalized eigenvector v belonging to e_4 so

$$(A-2I)v = e_4 \qquad (***)$$

But we don't have to solve any equations. Then matrix A is

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

The first column tells: $Ae_1 = 2e_1 + e_4$

or $(A-2I)e_1 = e_4$.

so $v = e_4$

The third column tells us $Ae_3 = e_2 + 2e_3$

or $(A-2I)e_3 = e_2$

so

$$u = e_3$$

Now we write down the Jordan basis $\mathcal{B}$. As we saw in the previous example we have to write in the correct order where the generalized eigenvector belonging to a eigenvector follows that eigenvector.

Here $\mathcal{B} = \{e_2, e_3, e_4, e_1\}$ so

$$P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since P is orthogonal we have

$$P = {}^tP^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now we check

$$P A P^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$ yes