

Lecture 2: Spanning Sets and Independent Sets

Definition

Let v_1, v_2, \dots, v_n be the vectors in a vector space V . A vector $u \in V$ is said to be a linear combination of v_1, v_2, \dots, v_n if there exist scalars c_1, c_2, \dots, c_n such that

$$u = c_1v_1 + c_2v_2 + \dots + c_nv_n.$$

The set of all linear combinations of v_1, v_2, \dots, v_n is said to be the **span** of v_1, v_2, \dots, v_n written

$$S(v_1, v_2, \dots, v_n) \text{ or } \text{span}(v_1, v_2, \dots, v_n)$$

- (1) The set of the solutions to the differential equation

$$\frac{d^2y}{dx^2} - y = 0 \quad (*)$$

is a vector space V under the rules of $+$ and \bullet for functions, Example II of the last lecture.

Then this vector space is spanned by e^x and e^{-x} . If y is a solution of $(*)$, then there are constants (scalars) $c_1, c_2 \in \mathbb{R}$ such that

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

Examples (continued)

- (2) $V = \mathbb{R}^3$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Then $S(e_1, e_2) =$ the xy -plane .

Suppose X is a set with a binary operation \circ so given any pair $x_1, x_2 \in X$ we have $x_1 \circ x_2 \in X$.

Let Y be a subset of X . We say Y is **closed** under \circ if for any pair $y_1, y_2 \in Y$ we have $y_1 \circ y_2 \in Y$.

Definition

Let $(V, +, \bullet)$ be a vector space and $U \subset V$ be a subset. Then U is said to be a **subvector space** or **subspace** of V if

- (i) U is closed under $+$,
- (ii) U is closed under \bullet .

Proposition

If U is closed under $+$ and \bullet , then U equipped with the restrictions of $+$ and \bullet satisfies the 8 vector space axioms $A1, A2, A3, A4$ and $S1, S2, S3, S4$. Hence $(U, +, \bullet)$ is a vector space. U is said to be a subspace of V .

Examples

- (1) The xy -plane $\subset \mathbb{R}^3$.
- (2) The space of polynomial functions of one variable of degree n
 $\text{Pol}_n(\mathbb{R}) \in \mathcal{F}_{\mathbb{R}}(\mathbb{R})$.

Spanning Sets

Proposition

Suppose v_1, v_2, \dots, v_n are vectors in V . Then $S(v_1, v_2, \dots, v_n)$ is a subspace of V .

Proof.

S is closed under $+$

$$\begin{aligned}c_1v_1 + c_2v_2 + \dots + c_nv_n + d_1v_1 + d_2v_2 + \dots + d_nv_n &= \\= (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \dots + (c_n + d_n)v_n.\end{aligned}$$

S is closed under \bullet

$$c(c_1v_1 + c_2v_2 + \dots + c_nv_n) = (cc_1)v_1 + (cc_2)v_2 + \dots + (cc_n)v_n. \quad \square$$

$S(v_1, v_2, \dots, v_n)$ is said to be the subspace of V spanned by v_1, v_2, \dots, v_n

Proposition

$S(v_1, v_2, \dots, v_n)$ is a subspace of V is the smallest subspace of V containing v_1, v_2, \dots, v_n .

Proof.

Suppose U contains v_1, v_2, \dots, v_n . Then since U is closed under $+$ and \bullet , any linear combination $c_1v_1 + c_2v_2 + \dots + c_nv_n$ must be in U . Hence

$$S(v_1, v_2, \dots, v_n) \subseteq U.$$

But U is the smallest subspace of V containing v_1, v_2, \dots, v_n , so

$$U \subseteq S(v_1, v_2, \dots, v_n),$$

so

$$U = S(v_1, v_2, \dots, v_n). \quad \square$$

Definition

Let V be a vector space. Then a collection of vectors $\{v_1, v_2, \dots, v_n\} \subset V$ is said to be a spanning set for V if

$$V = S(v_1, v_2, \dots, v_n).$$

Examples

- (1) (e_1, e_2, e_3) is a spanning set for \mathbb{R}^3 .
- (2) (e_1, e_2, \dots, e_n) is a spanning set for \mathbb{R}^n .
- (3) $(1, x, x^2, \dots, x^n)$ is a spanning set for $\text{Pol}_n(\mathbb{R})$.

Linear Independence

There are inefficient (too big) spanning sets for a vector space V . For example $\{(1, 0), (0, 1), (1, 1)\}$ is a spanning set for \mathbb{R}^2 but any two of the three vectors still spans.

Dependence Relation

Let $v_1, v_2, \dots, v_n \in V$. Then a dependence relation between v_1, v_2, \dots, v_n is an equation

$$c_1v_1 + c_2v_2 + \dots + c_nv_n = 0, \quad c_1, c_2, \dots, c_n \in \mathbb{R}.$$

The dependence relation is said to be the trivial dependence relation if all the c_i 's are zero. So in the above

$$1 \bullet (1, 0) + 1 \bullet (0, 1) - 1 \bullet (1, 1) = 0.$$

is a dependence relation.

Linear Independence

Definition

- (1) If v_1, v_2, \dots, v_n satisfy a nontrivial dependence relation then they are said to be linearly **dependent**.
- (2) v_1, v_2, \dots, v_n are said to be linearly **independent** if they are not linearly dependent.

So, v_1, v_2, \dots, v_n are linearly independent if

$$c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$$

\implies all the c_i 's are zero.

Exercise: Show e_1, e_2, \dots, e_n are linearly independent in \mathbb{R}^n .