Lecture 14 : The Gamma Distribution and its Relatives
The gamma distribution is a continuous distribution depending on two parameters, $\alpha$ and $\beta$. It gives rise to three special cases

1. **The exponential distribution** ($\alpha = 1, \beta = \frac{1}{\lambda}$)

2. **The $r$-Erlang distribution** ($\alpha = r, \beta = \frac{1}{\lambda}$)

3. **The chi-squared distribution** ($\alpha = \frac{\nu}{2}, \beta = 2$)
The Gamma Distribution

**Definition**

A continuous random variable $X$ is said to have gamma distribution with parameters $\alpha$ and $\beta$, both positive, if

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

What is $\Gamma(\alpha)$?

$\Gamma(\alpha)$ is the gamma function, one of the most important and common functions in advanced mathematics. If $\alpha$ is a positive integer $n$ then

$$\Gamma(n) = (n - 1)!$$

(see page 17)
Definition (Cont.)

So $\Gamma(\alpha)$ is an interpolation of the factorial function to all real numbers.

$\lim_{\alpha \to 0} \Gamma(\alpha) = \infty$

Graph of $\Gamma(\alpha)$

“$\Gamma(0) = \infty$”

$\Gamma(1) = 0!$

$\Gamma(2) = 1!$

$\Gamma(3) = 2!$

$\Gamma(4) = 3!$
I will say more about the gamma function later. It isn’t that important for Stat 400, here it is just a constant chosen so that

$$\int_{-\infty}^{\infty} f(x) \, dx = 1$$

The key point of the gamma distribution is that it is of the form

$$(\text{constant}) \ (\text{power of } x) \ e^{-cx}, \ c > 0.$$ 

The $r$-Erlang distribution from Lecture 13 is almost the most general gamma distribution.
The only special feature here is that $\alpha$ is a whole number $r$. Also $\beta = \frac{1}{\lambda}$ where $\lambda$ is the Poisson constant.

**Comparison Gamma distribution**

\[
\left(\frac{1}{\beta}\right)^\alpha \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}
\]

**r-Erlang distribution** $\alpha = r, \beta = \frac{1}{\lambda}$

\[
\lambda^r \frac{1}{(r-1)!} x^{r-1} e^{-\lambda x}
\]
### Proposition

Suppose $X$ has gamma distribution with parameters $\alpha$ and $\beta$ then

1. $E(X) = \alpha \beta$
2. $V(X) = \alpha \beta^2$

so for the $r$-Erlang distribution

1. $E(X) = \frac{r}{\lambda}$
2. $V(X) = \frac{r}{\lambda^2}$
Proposition (Cont.)

As in the case of the normal distribution we can compute general gamma probabilities by standardizing.

Definition

A gamma distribution is said to be standard if $\beta = 1$. Hence the pdf of the standard gamma distribution is

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The cdf of the standard
Definition (Cont.)

Gamma function is called the incomplete gamma function (divided by $\Gamma(\alpha)$)

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^x x^{\alpha-1} e^{-x} \, dx$$

(see page 13 for the actual gamma function)

It is tabulated in the text Table A.4 for some (integral values of $\alpha$)

Proposition

Suppose $X$ has gamma distribution with parameters $\alpha$ and $\beta$. Then $Y = \frac{X}{\beta}$ has standard gamma distribution.
Proof.

We can prove this, \( Y = \frac{X}{\beta} \) so \( X = \beta y \).

Now \( f_X(x)dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx \).

Now substitute \( x = \beta y \) to get

\[
f_Y(y)dy = \frac{1}{\beta^\alpha} \frac{1}{\Gamma(\alpha)} (\beta y)^{\alpha-1} e^{-\beta y/\beta} d(\beta y)
\]

\[
= \frac{1}{\beta^\alpha} \frac{1}{\Gamma(\alpha)} \beta^\alpha y^{\alpha-1} e^{-y} \beta dy
\]

\[
= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-t} dy
\]

standard gamma
Example 4.24 (cut down)
Suppose $X$ has gamma distribution with parameters $\alpha = 8$ and $\beta = 15$. Compute

$$P(60 \leq X \leq 120)$$

Solution

*Standardize, divide EVERYTHING by $\beta = 15$.*

$$P(60 \leq X \leq 120) = P\left(\frac{60}{15} \leq \frac{X}{15} \leq \frac{120}{15}\right) = P(4 \leq Y \leq 8) = F(8) - F(4)$$

*from table A.4*

$$= .547 - .051 = .496$$
The Chi-Squared Distribution

Definition

Let \( \nu \) (Greek letter nu) be a positive real number. A continuous random variable \( X \) is said to have chi-squared distribution with \( \nu \) degrees of freedom if \( X \) has gamma distribution with \( \alpha = \nu/2 \) and \( \beta = 2 \). Hence

\[
    f(x) = \begin{cases} 
        \frac{1}{2^{\nu/2}} \Gamma(\nu/2) x^{\nu/2-1} e^{-x/2}, & x > 0 \\
        0, & \text{otherwise}.
    \end{cases}
\]

We will write \( X \sim \chi^2(\nu) \).
The reason the chi-squared distribution is that if

\[ Z \sim N(0, 1) \quad \text{then} \quad X = Z^2 \sim \chi^2(1) \]

and if \( Z_1, Z_2, \ldots, Z_m \) are independent random variables the

\[ Z_1^2 + Z_2^2 + \cdots + Z_m^2 \sim \chi^2(m) \]

(later).

Proposition (Special case of pg. 6)

If \( X \sim \chi^2(\nu) \) then

(i) \( E(X) = \nu \)

(ii) \( V(X) = 2\nu \)
Appendix : The Gamma Function

Definition

For $\alpha > 0$, the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x} \, dx$$

Remark 1

It is more natural to write

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x} \frac{dx}{x}$$

but I won’t explain why unless you ask.
Remark 2

In the complete gamma function we integrate from 0 to infinity whereas for the incomplete gamma function we integrate from 0 to $x$.

$$F(x; \alpha) = \int_0^x y^{\alpha-1} e^{-y} \, dx.$$ 

Thus

$$\lim_{x \to \infty} F(x; \alpha) = \Gamma(\alpha).$$
Remark 3

Many of the “special functions” of advanced mathematics and physics e.g. Bessel functions, hypergeometric functions... arise by taking an elementary function of x depending on a parameter (or parameters) and integrating with respect to x leaving a function of the parameter. Here the elementary function is $x^{\alpha-1}e^{-x}$. We “integrate out the x” leaving a function of $\alpha$. 
**Lemma**

\[ \Gamma(1) = 1 \]

**Proof.**

\[ \Gamma(1) = \int_0^\infty e^{-x} \, dx = (e^{-x}) \bigg|_0^\infty = 1 \]

**The Functional Equation for the Gamma Function**

**Theorem**

\[ \Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \quad \alpha > 0 \]

**Proof.**

Integrate by parts

\[
\Gamma(\alpha + 1) = \int_0^\infty \underbrace{x^{-\alpha}}_u \underbrace{e^{-x}}_v \, dx = (-x^{\alpha} e^{-x}) \bigg|_0^\infty - \int_0^\infty \frac{du}{\alpha x^{\alpha-1}} (-e^{-x}) \, dx
\]

\[
= \alpha \int_0^\infty x^{\alpha-1} e^{-x} \, dx
\]

\[\square\]

Lecture 14 : The Gamma Distribution and its Relatives
Corollary

If $n$ is a whole number

$$\Gamma(n) = (n - 1)!$$

Proof.

I will show you $\Gamma(4) = 3Q$

$$\Gamma(4) = \Gamma(3 + 1) = 3\Gamma(3)$$

$$= 3\Gamma(2 + 1) = (3)(2)\Gamma(2)$$

$$= (3)(2)\Gamma(1 + 1) = (3)(2)(1)F(1)$$

$$= (3)(2)(1)$$

In general you use induction.

We will need $\Gamma$ (half integers) e.g. $\Gamma\left(\frac{5}{2}\right)$.

Theorem

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
I won’t prove this. Try it.

\[ \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \]

\[ \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2} + 1\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \sqrt{\pi} \]

In general

\[ \Gamma\left(\frac{2n + 1}{2}\right) = \frac{(1)(3)(5) \ldots (2n - 1)}{2^n} \sqrt{\pi} \]

For statistics we will need only \( \Gamma(\text{integer}) = (\text{integer}-1)! \)

and \( \Gamma\left(\frac{\text{add integer}}{2}\right) = \text{above} \)